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# An Analysis of the Telegrapher Equation with a Bifurcation Parameter to Model Relativistic Diffusion

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by

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*"Just do yourself a little addsy-subtractsy"*

Dr. Brian Pigott

# Abstract

In this paper, we derive a solution to the telegrapher equation. We then apply a bifurcation parameter to the telegrapher equation in order to analyze the behavior of the solution as it changes classification. In order to obtain the solution to both the telegrapher and modified telegrapher equation, we derive the heat equation and telegrapher equation using a continuous random walk. We also solve the heat equation using invariant properties of a particular solution, a random walk analysis, and a Fourier-Laplace transform. The solution to the telegrapher equation contains modified Bessel functions, so we also derive the solutions to both the Bessel and modified Bessel equation. Lastly, we rigorously obtain a solution to the telegrapher equation with an added bifurcation parameter. This solution represents a complete distribution of the solution to the standard telegrapher equation as its solutions transition between classifications.

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*To my friends, family, and professors...*

# Chapter 1

# Introduction

## 1.1 Motivation

There have been numerous works concerning the topic of a relativistic heat equation.[1–3]. In order to understand the relevance of this work, we must first note that the standard heat equation is of the form:

$$U_t - kU_{xx} = 0, \tag{1.1}$$

where $U$ is a function of $t$ and $x$ and $k$ is a positive constant related to the diffusion of heat through an object. The solution of this partial differential equation is a Gaussian of the following form:

$$U(t,x) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-\frac{(x-y)^2}{4kt}} dy, \tag{1.2}$$

where $\phi(y)$ is the initial temperature distribution at $t = 0$. Later in this paper, we will show multiple ways of deriving this solution. In this solution, we can see that at any nonzero value of $t$, there is a defined value of $U(t,x)$ at all values of $x$. This is a result that does not agree with special relativity. Special relativity states that no information can travel faster than the speed of light; however, as we just stated, (1.2) implies that at any value of time, heat can be found at a distance infinitely far from the origin. Even with this non-relativistic behavior, the standard heat equation is still a well-accepted and accurate model of the diffusion of heat through an object. Thus, an abundance of work has been done attempting to find a modification of the heat equation that preserves the accuracy of the heat equation, while yeilding a relativistic solution on the infinite line.

The standard heat equation and its solution can be derived in multiple ways. In this analysis, we will derive the heat equation using a physical derivation of heat flowing through a long straight rod following [4] as well as a derivation using a single-step continuous random walk following [5]. The probability distribution of finding a random walker at some point $x$ after $t$ seconds is equivalent to the solution of the heat equation. We can use a similar method to derive the telegrapher equation. The telegrapher equation is a finite-velocity bound version of the heat equation, derived from a multistep random walk [1].

$$U_{tt} + (1/T)U_t - v^2 U_{xx} = 0. \tag{1.3}$$

In the multistage random walk we track the momentum of the walker as opposed to the random walk derivation of the heat equation where we find a probability distribution of the location of the walker [1] . While the solution to the telegrapher equation is a relativistic representation of the heat equation solution, work has been done that suggests the telegrapher solution differs in certain relativistic scenarios from those of the heat equation.[2]

When solving the Telegrapher Equation, we find that modified Bessel functions are involved in the solution. In order to understand the solution to the Telegrapher equation, we will analyze the derivation and behavior of these Bessel functions later in this paper. Also, we must perform a Fourier-Laplace transform in order to solve the equation, so that will be covered extensively throughout this analysis.

Lastly, in an attempt to modify the telegrapher equation to better model a relativistic heat equation, we will introduce a bifurcation parameter into the equation. This parameter, $\lambda$ will be placed as follows:

$$\lambda U_{tt} + (1/T)U_t - v^2 U_{xx} = 0. \tag{1.4}$$

We call this a bifurcation parameter because as the value of the parameter transitions from positive to zero, the classification of the partial differential equation transitions from hyperbolic to parabolic. This classification comes from The Classification Theorem of second order partial differential equations [4]. We will further discuss this concept in Chapters 8 and 9. In short, this transition will cause the solutions of the equation to transition from behaving like that of a wave equation to that of the heat equation, which is characteristic of a relativistic heat solution. In Chapter 9, we will solve the modified telegrapher equation with this bifurcation parameter in place and examine the effects that it has on the ordinary solution to the telegrapher equation.

# Chapter 2

# Physical Derivation of the Heat Equation

## 2.1 Heat Equation on an Infinite Length Rod

Consider a wire or rod that is perfectly insulated except possibly at the ends. We assume that temperature must remain constant on a cross section of the wire. Our list of variables will be as follows:

- $u(x,t)$ = the heat in the rod at position $x$ and time $t$.
- D = density of the rod
- C = specific heat of the rod
- L = length of the rod
- A = area of cross-section

Consider a small chunk of the rod at $x = x_0$ with thickness of $\Delta x$. The mass of the chunk is $DA\Delta x$. Thus the energy required to raise the temperature of the slab from 0 to $u(x_0,t)$ is $(\Delta T)CM = u(x_0,t)CDA\Delta x$. If we let $\Delta x \to 0$, and add up the energies between arbitrary points a and b, we get the following expression for the heat energy from $a$ to $b$ at time $t$:

$$E(t) = \int_a^b CDAu(x,t)dx. \tag{2.1}$$

Heat flows from hotter regions to colder regions at a rate that is proportional to the temperature difference divided by the distance between the regions. Quantitatively, the rate at which energy passes through the cross section at $x = a$ in the positive direction is:

$$-kA(\frac{\partial u}{\partial x})\Big|_{x=a} = -kAu_x(a,t) \tag{2.2}$$

where $k$ is known as the thermal conductivity of the object. If $u_x(a,t) < 0$, then the temperature to the left of $x = a$ is greater than that of the right of $x = a$. The net rate that the heat enters a cross section is the opposite the rate that it exits the other end of the cross section. Thus, we can now write an expression for the thermal energy as:

$$\begin{aligned} E'(t) &= -kAu_x(a,t) + kAu_x(b,t) && (2.3)\\ &= kAu_x(x,t)|_a^b && (2.4)\\ &= \int_a^b kA\frac{\partial}{\partial x}u_x(x,t)dx && (2.5) \end{aligned}$$

The step from (2.4) to (2.5) came from implementing the Fundamental Theorem of Calculus. We can also obtain $E'(t)$ by differentiating with respect to $t$ in (2.1):

$$E'(t) = \int_a^b CDAu_t(x,t)dx \tag{2.6}$$

and thus,

$$\int_a^b CDAu_t(x,t)dx = E'(t) = \int_a^b kAu_{xx}(x,t)dx \tag{2.7}$$

Now we can divide by CDA on both sides and define $\kappa = \frac{k}{CD}$ we get the following expression:

$$\int_a^b [u_t - \kappa u_{xx}]dx = 0. \tag{2.8}$$

Since a and b are arbitrary points, we can drop this integration and obtain:

$$u_t - \kappa u_{xx} = 0. \tag{2.9}$$

Equation (2.9) is referred to as the heat equation.

# Chapter 3

# Solution of The Heat Equation Using Invariance Properties

## 3.1 Introduction

In this section, we will solve the heat equation using a list of invariance properties that we know about the solution. The process of finding this solution follows the work of Strauss [6]. We will solve the standard heat equation found in (1.1) and derived in the previous chapter. Note that the equation is of the form:

$$u_t = ku_{xx} \tag{3.1}$$

$$u(x, 0) = \phi(x)$$

$$-\infty < x < \infty$$

We look to solve for a particular $\phi(x)$ and build a general solution from there. We'll use these five invariance properties:

- The translate $u(x - y, t)$ of any solution $u(x, t)$ is another solution, for any fixed $y$.
- Any derivative of a solution is a solution.
- A linear combination of solutions is a solution.
- An integral of a solution is a solution. Thus if $S(x, t)$ is a solution, then so is $S(x - y, t)$ and so is
$$v(x, t) = \int_{-\infty}^{\infty} S(x - y, t)g(y)dy.$$
for any $g(y)$, as long as the improper integral converges.

- If $u(x,t)$ is a solution, so is the dialated function $u(\sqrt{a}x, at)$, for any $a > 0$.

Our goal is to find a particular solution and then use the fourth bullet point to construct all of the other solutions. The particular solution we will look for is $Q(x,t)$ which satisfies:

$$Q(x,0) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{3.2}$$

## 3.2 Behavior of a chosen specific solution

Here we seek $Q(x,t)$ as a solution of $Q_t = kQ_{xx}.Q(x,t)$ will be of the special form $Q(x,t) = g(p)$ where $p = \frac{x}{\sqrt{4kt}}$ and $g$ is a function of one variable.

We expect this form because the fifth bullet point says that the equation is invariant under dilation $x \to \sqrt{a}x, t \to at$. $Q(x,t)$ should not respond to the dilation either. This can only happen if $Q$ depends on $x$ and $t$ solely through the combination $\frac{x}{\sqrt{t}}$. Thus let $p = \frac{x}{\sqrt{4kt}}$ and look for $Q(x,t)$ to satisfy the fifth bullet point conditions mentioned.

Using $Q(x,t) = g(p), p = \frac{x}{4kt}$, we can convert the heat equation into an ordinary differential equation (ODE) in terms of $g$ using the chain rule:

$$Q_t = \frac{\partial g}{\partial p}\frac{\partial p}{\partial t} = -\frac{1}{2}\frac{x}{\sqrt{4k}}\frac{1}{\sqrt{t^3}} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p) \tag{3.3}$$

$$Q_x = \frac{\partial g}{\partial p}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p) \tag{3.4}$$

$$Q_{xx} = \frac{\partial Q_x}{\partial p}\frac{p}{\partial x} = \frac{1}{4kt}g''(p) \tag{3.5}$$

Thus,

$$0 = Q_t - kQ_{xx} = \frac{1}{t}[-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)] = g'' + 2pg' \tag{3.6}$$

This is a linear 2nd order ODE, however the equation is first order in $g'(p)$. Thus we can solve for $g'(p)$ by using an integrating factor. We can then integrate the expression for $g'(p)$ to find $g(p)$. The result of this calculation is as follows:

$$Q(x,t) = C_1 \int e^{-p^2}dp + C_2. \tag{3.7}$$

## 3.3 Solving the heat equation in the form of a second order ODE

Now we look to find a completely explicit formula for Q. We've just shown that:

$$Q(x,t) = C_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp + C_2 \tag{3.8}$$

Note that this solution is only valid when $t > 0$. When $t < 0$, we note (3.2) establishes that $Q(x, 0) = 0$. Now we note two facts:
(1) If $x > 0$,

$$1 = \lim_{t\to 0} Q(x,t) = C_1 \int_0^{\infty} e^{-p^2} dp + C_2 = C_1 \frac{\sqrt{\pi}}{2} + C_2 \tag{3.9}$$

(2) If $x < 0$,

$$0 = \lim_{t\to 0} Q = C_1 \int_0^{-\infty} e^{-p^2} dp + C_2 = -C_1 \frac{\sqrt{\pi}}{2} + C_2 \tag{3.10}$$

We can determine the constants by solving the system of equations. When we do this we see that

$$C_1 = \frac{1}{\sqrt{\pi}}, C_2 = \frac{1}{2}.$$

We can now plug this into (7) and obtain:

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp \tag{3.11}$$

## 3.4 Defining the heat equation for a generalized set of initial conditions

After finding Q, we now define $S = \frac{\partial Q}{\partial x}$. By the second invariant property, $S$ is also a solution. Given any function $\phi$, we also define:

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y) dy \tag{3.12}$$

for all $t > 0$. By the fourth invariant property, $u$ is another solution. We claim that $u$ is the unique solution of the heat equation with initial condition $u(x, 0) - \phi(x)$. We verify this in the following way:

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)dy \\ &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y} Q(x-y,t)\phi(y)dy \\ &= \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)dy - Q(x-y),t)\phi(y)|_{y=-\infty}^{y=\infty}. \end{aligned}$$

Assume $\phi(y) = 0$ for large $|y|$. We then integrate this by parts and obtain:

$$u(x,0) = \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)dy \tag{3.13}$$

$$= \int_{-\infty}^{x} \phi'(y)dy = \phi|_{-\infty}^{x} = \phi(x) \tag{3.14}$$

This is due to the assumption that $\lim_{x\to-\infty} Q(x,t) = 0$, and thus $\lim_{x\to-\infty} \phi(x) = 0$. Notice that:

$$S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi k t}} e^{\frac{-x^2}{4kt}}, \tag{3.15}$$

for all $t > 0$. Thus we conclude the solution to the heat equation to be:

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} \phi(y)dy. \tag{3.16}$$

# Chapter 4

# Finding the solution to the Heat Equation using random walks

## 4.1 Introduction

In this chapter, we will use a simple, one dimensional random walk in order to derive a solution to the heat equation using the process outlined by [5]. Here we consider a random walker that has equal probability of moving one step in the positive $x$ direction and moving one step in the negative $x$ direction. Given the initial position of the walker, if we analyze a distribution of the probability of the random walker being at some position at a time $t$ and then take the limit as $t \to \infty$, then we obtain a distribution that is the same as the solution to the heat equation.

## 4.2 Finding the expectation value and variance of the position of a walker at some time

As previously mentioned, we will begin this derivation to the solution of the heat equation by analyzing the behavior of a random walker. In this analysis, we will denote the position of the walker after $n$ steps as $S_n$ and we will denote the change in position from the previous step as $x_j$, where $j$ is a specific step of the random walk. We note that:

$$P\{x_j = 1\} = P\{x_j = -1\} = \frac{1}{2}. \tag{4.1}$$

Here $P\{x_j = 1\}$ and $P\{x_j = -1\}$ denotes the probability that the walker moves one unit in the positive direction or negative direction on step $j$ of the walk. In other words, the walker has the same chance of moving one step in the positive direction as it does moving one step in the negative direction. We will also let one step in the random walk represent one unit of time. This means that a time of $t = 1$ corresponds to one step of the random walk. Thus, we can calculate the position, $S_n$ of the walker by summing the change in position, $x_j$ of each step. That is, the position of the walker at some time $t = n$ is calculated as follows:

$$S_n = x_0 + x_1 + ... + x_n. \tag{4.2}$$

We will now compute the expected value of the final position and total distance traveled away from the origin at $t = n$ for a random walker. In order to do this, we note that the calculation for expectation of some variable, denoted $E(A)$ is,

$$E(A) = \sum_z zP\{A = z\}, \tag{4.3}$$

where $P\{A = z\}$ is the probability of the variable $A$ having the value $z$.
Thus, in the case of the random walker, we calculate the expectation value of the position of the random walk as follows:

$$E(S_n) = (1)\frac{1}{2} + (-1)\frac{1}{2} = 0, \tag{4.4}$$

which would make sense because the walker has the same probability of moving one step in the positive direction as moving one step in the positive direction. Thus, we would expect the positive steps and negative steps to cancel out after a large number of steps. If we want to calculate the expected distance that the walker is away from its initial position, we would need to calculate $E(\mid S_n \mid)$. However, it is easier for us to calculate $E(S_n^2)$, which is the expectation value of the distance from the starting point squared, and then simply take the square root of this value. This will not give us the $E(\mid S_n \mid)$ exactly, but it will give us the typical distance that the walker is from the origin expressed in terms of total steps taken. Calculating this value is done as follows:

$$E(S_n^2) = E\Big[\Big(\sum_{j=1}^{n} x_j\Big)^2\Big] \tag{4.5}$$

$$= E\Big[\sum_{k=1}^{n}\sum_{j=1} x_j x_k\Big]$$

$$= E\Big[\sum_{j=k} x_j x_k\Big] + E\Big[\sum_{j\neq k} x_j x_k\Big]. \tag{4.6}$$

Now, we note that when $j = k$ we know that $x_j = x_k$. Since $x_j = x_k = \pm 1$, this means that $x_j x_k = 1$. Thus, $\sum_{j=k}^{n} x_j x_k = n$ at all values of $j$ and $k$. Using our definition

in (4.3), we see that the only possible value of $z$ is $z = n$ and $P\{\sum_{j=k}^{n} x_j x_k = n\} = 1$. Therefore we know that $E\Big[\sum_{j=k}^{n} x_j x_k\Big] = n$.
When we look at the second term of (4.6), using similar logic, we see that $x_j x_i = \pm 1$, and $P\{x_j x_k = 1\} = P\{x_j x_k = -1\} = \frac{1}{2}$. Thus, $E\Big[\sum_{j \neq k}^{n} x_j x_k\Big] = (1)\frac{1}{2} + (-1)\frac{1}{2} = 0$. Therefore, we can simplify (4.6) to the following expression:

$$E(S_n^2) = n \tag{4.7}$$

If we take the square root of this result, we can obtain an expression for the expected distance that the walker is from the origin after $n$ steps. Thus,

$$E(\mid S_n \mid) = \pm\sqrt{n} \tag{4.8}$$

This means that as the number of steps the walker takes increases, the probability of finding the random walker at the origin will decrease by a factor of $\sqrt{n}$. This is not an exact probability distribution, but this gives a sense of the behavior of the walker as the amount of time steps increases.

## 4.3 Finding a probability distribution for the position of a random walker at time $t = n$

We note that since $x_j = \pm 1$, if the walker has taken an even number of steps, then the value of $S_n$ must also be even. Similarly, we know that if the walker has taken an odd number of steps, then the value of $S_n$ must also be odd. Without loss of generality, we can assume that the walker has taken an even number of steps. We can do this because we can simply think of the walker taking two independent steps at once, then we would record the position. This would always put the walker at an even value of $S_n$ and would also allow the walker to return to its initial position.
Given that the walker is at a position of $2j$ and has taken $2n$ steps, which guarantees the amount of steps taken is even, this means that the walker must take $n + j$ steps in the positive direction and $n - j$ in the negative direction. We also know that the probability of the walker moving in a particular direction is $\frac{1}{2}$. Thus we could calculate the probability of the walker getting to the specific value of $2j$ by multiplying the total ways of choosing $2n$ steps, given there must be $n+j$ total steps in the positive direction, by the probability of one specific path with $2n$ steps. We note that the probability of one specific path of the random walker is simply the probability of one specific step, $\frac{1}{2}$ multiplied by total steps $2n$. Thus we can write an expression for finding the walker at

a specific, even integer valued location as follows:

$$P\{S_{2n} = 2j\} = \binom{2n}{n+j} 2^{-2n} = \frac{(2n)!}{(n+j)!(n-j)!} 2^{-2n}. \tag{4.9}$$

Now we will use Sterling's Equation, which states that:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}} = 1. \tag{4.10}$$

To expand the value of $P\{S_{2n} = 2j\}$, we will use the Stirling approximation formula. When we do this, we obtain the following expression:

$$P\{S_{2n} = 2j\} = \frac{\sqrt{2}}{\sqrt{2\pi}} \Big(1 - \frac{j^2}{n^2}\Big)^{-n} \Big(1 + \frac{j}{n}\Big)^{-j} \Big(1 - \frac{j}{n}\Big)^{j} \Big(\frac{n}{n^2 - j^2}\Big)^{\frac{1}{2}} \tag{4.11}$$

for very large values of $n$.

In the previous section, we explained that the average distance from the origin was on the order of $\sqrt{n}$. Thus if we replace $j$ with $r\sqrt{n}$ for some value $r$, we would obtain an expression for the probability that the random walker is some multiple of the expected distance from the origin. When we make this substitution we obtain:

$$\begin{aligned} P\{S_2n = 2r\sqrt{n}\} &= \frac{\sqrt{2}}{\sqrt{2\pi n}} \Big(1 - \frac{r^2}{n}\Big)^{-n} \Big[\Big(1 + \frac{r}{\sqrt{n}}\Big)^{-\sqrt{n}}\Big]^{r} \\ &\quad \times \Big[\Big(1 - \frac{r}{\sqrt{n}}^{-\sqrt{n}}\Big)\Big]^{-r} \Big(\frac{1}{1 - \frac{r^2}{n}}\Big)^{\frac{1}{2}}. \end{aligned} \tag{4.12}$$

Now we will take the limit of this equation as $n \to \infty$. In order to do this, we will use the well-known limit,

$$\lim_{n\to\infty} \Big(1 + \frac{a}{n}\Big)^n \to e^a. \tag{4.13}$$

Thus, (4.12) simplifies to:

$$P\{S_2n = 2r\sqrt{n}\} = \frac{\sqrt{2}}{\sqrt{2\pi n}} e^{r^2} e^{-r^2} e^{-r^2}.$$

Since $j = r\sqrt{n}$,

$$P\{S_2n = 2r\sqrt{n}\} = \frac{1}{\sqrt{\pi n}} e^{\frac{-j^2}{n}}. \tag{4.14}$$

Now, we note that we have been treating $n$ in the same manor in which we would treat a time, $t$ variable. Also, we have been treating $j$ just as we would treat an $x$ value. However, we note that $x$ is one half the size of $j$. Thus, we replace $j = \frac{x}{2}$ and can simply replace $n$ with $t$ and obtain:

$$P(t,x) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \tag{4.15}$$

However, since the integral of this Gaussian over all space $x$ is equal to $\sqrt{2}$, and we would want the probability of finding the walker over all space to be equal to 1, we must multiply by a normalizing factor of $\frac{1}{\sqrt{2}}$. When this is done, we obtain the following solution:

$$P(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \tag{4.16}$$

We can see that this is the same solution that we found using the invariance properties in Chapter 3. Thus, here we have derived a solution to the standard heat equation found in (1.1) by analyzing the behavior of a discrete random walk. Next, we will find a solution to the standard heat equation using a Fourier-Laplace transform, which will be an essential technique in solving the telegrapher equation.

# Chapter 5

# Solving the Heat Equation using the Fourier-Laplace Transform

## 5.1 Introduction

In Chapter Two, we derived the heat equation using a physical analysis. We then proceeded to solve the heat equation by means of invariance properties and a random walk analysis in chapters two and three, respectively. However, when solving the telegrapher equation, we need to use a Fourier-Laplace transform. The Fourier-Laplace transform takes the heat equation, which is a second order partial differential equation, and transforms it into an algebraic expression. This is done by rewriting the equation in frequency space by using the Fourier transform, then rewriting the equation in terms of exponential terms.

We note that the Fourier Transform of a given function $f(t,x)$ with respect $x$ is as follows:

$$\hat{f}(t,\omega) = \int_{-\infty}^{\infty} f(t,x)e^{i\omega x}dx. \tag{5.1}$$

For simplicity, we will denote the Fourier transform of the function $U$ as $\mathcal{F}[U]$. Also, we note that the Laplace Transform of a given function $f(t,x)$ with respect to $t$ is given by:

$$\bar{f}(s,x) = \int_{0}^{\infty} f(t,x)e^{-st}dt. \tag{5.2}$$

Similar to the Fourier transform, we will denote the Laplace transform of $U$ as $\mathcal{L}[U]$. Thus, we can combine these two transforms to obtain the Fourier-Laplace transform that will be of the following form:

$$\tilde{f}(s,\omega) = \int_0^{\infty} e^{-st} \int_{-\infty}^{\infty} f(t,x) e^{i\omega x} dx dt. \tag{5.3}$$

As previously mentioned, after we perform the Fourier-Laplace transform, we will have an algebraic equation that we can solve for the solution of the heat equation in Fourier-Laplace space. Once we have solved for this solution, we will then perform an inverse Fourier-Laplace transform to obtain the solution in typical $t, x$ space.

We note that the inverse Fourier transform of some function $f(s,\omega)$ with respect to $\omega$ is as follows:

$$f(s,x) = \int_{-\infty}^{\infty} \hat{f}(s,\omega) e^{-i\omega x} d\omega \tag{5.4}$$

We will denote the inverse Fourier transform of $U$ as $\mathcal{F}^{-1}[U]$. The inverse Laplace transform is more complicated, as it involves a contour integral in the complex plane. We will use tables of well-known inverse Laplace transforms in order to perform this calculation [7]. We denote the inverse Laplace transform as follows:

$$f(t,x) = \mathcal{L}^{-1} \bar{f}(s,x) \tag{5.5}$$

Thus, the inverse Fourier-Laplace double transform is done by the following calculation:

$$f(x,t) = \int_{-\infty}^{\infty} e^{-i\omega x} \mathcal{L}^{-1}[\tilde{f}(s,x)] d\omega \tag{5.6}$$

Now we will solve the standard heat equation with this outlined technique.

## 5.2 The Heat Equation, Initial Conditions, and the Fourier-Laplace Transform of the Initial Conditions

As mentioned in the previous section, the standard heat equation is of the form:

$$U_t - aU_{xx} = 0, \tag{5.7}$$

where $a$ is a constant related to the diffusion of heat through a material and $U$ is a function of $t$ and $x$.
We will analyze the initial condition that correspond to a point heat source centered at

$x = 0$. Thus the initial condition we will use are is follows:

$$U(0, x) = \delta_0(x). \tag{5.8}$$

However, we will be performing a Fourier-Laplace transform on the heat equation, so we need to establish the initial conditions in both Fourier and Fourier-Laplace space. This means we must first perform a Fourier transform on the initial conditions. When we do this we obtain the initial conditions in Fourier space:

$$\hat{U}(0, \xi) = 1, \tag{5.9}$$

where $\xi$ is the spatial Fourier transform variable. Also note that in order to do this, we used the well known identity that the Fourier Transform of the delta function is 1.

Next, we will take the Laplace transform of the initial in (5.8), in order to obtain the initial conditions in Fourier-Laplace space:

$$\tilde{U}(0, \xi) = \int_0^\infty (1)e^{-st} dt = \frac{1}{s}, \tag{5.10}$$

where $s$ is the temporal Laplace transform variable. Now, we will apply these initial conditions in their respective transformed spaces to find a fundamental solution to the heat equation on an infinite line.

## 5.3 Performing the Fourier Transform on the Standard Heat Equation

We can actually solve the Heat Equation by performing a Fourier Transform without a Laplace transform. Here, we will show how we can solve the heat equation with only the Fourier transform, and then show how we can solve this using the Fourier-Laplace double transform.

Now, note that when we apply the Fourier Transform to the standard Heat Equation, we get the following result:

$$\int_{-\infty}^{\infty} [U_t - aU_{xx}] e^{i\omega x} dx = 0 \tag{5.11}$$

When we perform this transform, we use two well know properties of the Fourier Transform [7]:

$$\mathcal{F}[U_t](x) = \hat{U}_t \tag{5.12}$$

$$\mathcal{F}[U_x](x) = (i\xi)\hat{U} \tag{5.13}$$

where $\mathcal{F}[U](x) = \hat{U}$. Thus, when we apply these two properties to the Fourier Transform of the heat equation, we get the following result:

$$\hat{U}_t + a\xi^2\hat{U} = 0, \tag{5.14}$$

where $\hat{U}$ is a function of $(t, \xi)$.

After performing the Fourier Transform we can now take the Laplace Transform and obtain an algebraic expression for the solution of the equation. However, another way of obtaining a solution to the heat equation is by seeing that in Fourier space, the standard heat equation is now a linear, 1st order ordinary differential equation with respect to $t$. This means we could find the solution to the heat equation by finding the solution of this ordinary differential equation, and then perform an inverse Fourier Transform to obtain the solution in $(t, x)$ space. In order to find the solution to (5.14) we multiply each side of the equation by an integrating factor $\mu = e^{\xi^2 dt} = e^{-a\xi^2 t}$. When we do this we obtain:

$$\begin{aligned} &e^{-a\xi^2 t}\hat{U}_t - a\xi^2 e^{-a\xi^2 t}\hat{U} = 0 \\ &\frac{d}{dt}\left(e^{-a\xi^2 t}\hat{U}\right) = 0 \end{aligned} \tag{5.15}$$

Now we integrate both sides of (5.15) with respect to $t$:

$$e^{a\xi^2 t}\hat{U} = C(\xi)$$

Where $C(\xi)$ is a function that is constant with respect to $\xi$. It follows that:

$$\hat{U} = e^{-a\xi^2 t}C(\xi) \tag{5.16}$$

Now, we use the initial condition in Fourier space in (5.9) to solve for the arbitrary constant function:

$$\hat{U}(0, \xi) = 1 = e^0(C(\xi)) \rightarrow C(\xi) = 1$$

Thus, the solution of the heat equation in Fourier space is:

$$\hat{U}(t, \xi) = e^{-at\xi^2}. \tag{5.17}$$

Now we will perform the inverse Fourier transform on (5.17) in order to obtain the solution in $t, x$ space. In order to do this, we will use the well known identity [7] that the Fourier transform of a Gaussian in real space is a Gaussian in Fourier space with a normalizing factor of $2\sqrt{\pi}$. This applies here because we are taking the inverse Fourier transform of a Gaussian in Fourier space. Thus, the inverse Fourier transform of (5.17) can be expressed as follows:

$$U(t,x) = \frac{1}{\sqrt{4\pi at}} e^{\frac{-x^2}{4at}} \tag{5.18}$$

Note that this is the same solution that we obtained when deriving the solution to the heat equation using the random walk method in chapter 4. This is the process of solving the standard heat equation by only using a Fourier Transform as opposed to the Fourier-Laplace double transform.

## 5.4 Performing the Fourier-Laplace Transform on the Standard Heat Equation

When solving the Telegrapher Equation, we must use a Fourier-Laplace Transform, as opposed to only using a Fourier Transform . While we could use the method used in the previous section to solve (5.17), performing a Laplace Transform on the equation will give us a simple algebraic expression to solve, rather than a differential equation.

Thus, when we take the Laplace Transform of (5.17) we perform the following calculation:

$$\int_0^\infty [\hat{U}_t + a\xi^2 \hat{U}] e^{-st} dt = 0. \tag{5.19}$$

In order to perform this transform, we use the following identities of the Laplace Transform[7]:

$$\mathcal{L}[U_t](s) = s\tilde{U} - U(0) \tag{5.20}$$

$$\mathcal{L}[aU](s) = a\tilde{U} \tag{5.21}$$

where $\mathcal{L}[U](s) = \tilde{U}$ and $a$ is some constant with respect to the transformed variable. When we apply (5.20) to the first term of the equation and (5.21) to the second term of the equation we obtain the following equation in Fourier-Laplace space:

$$s\tilde{U} - 1 + a\xi^2 \tilde{U} = 0 \tag{5.22}$$

where Ũ is the solution to the heat equation in Fourier-Laplace space and is a function of $s, \xi$. Now we will solve this algebraic expression for Ũ and obtain:

$$\tilde{U} = \frac{1}{s + a\xi^2} \tag{5.23}$$

Now we will perform the inverse Laplace transform on (5.23) in order to obtain the solution in Fourier space. In order to perform this calculation, we will use the following identity of the Laplace Transform:

$$\mathcal{L}[e^{bt}](s) = \frac{1}{s-b}[7] \tag{5.24}$$

Our expression is equivalent to the property in (5.24) with $b = a\xi^2$. Thus the inverse Laplace Transform of (5.23) is evaluated as follows:

$$\hat{U} = e^{a\xi^2 t} \tag{5.25}$$

Now we note that this is the same expression we found in (5.18) when we solved the differential equation in Fourier space. Thus, when we take the inverse Fourier transform of (5.25), we obtain the same result as (5.18), which agrees with the solution that we found by examining the discrete random walk. This is the same technique we will use in order to solve the telegrapher equation. In the next chapter, we will analyze the Bessel Equation and its solutions in order to understand the solutions to the telegrapher equation in the later chapters.

# Chapter 6

# Bessel Functions

## 6.1 Introduction

When obtaining a solution to the telegrapher equation, we find that the solution contains modified Bessel functions. These Bessel functions are simply solutions to specific ordinary differential equations called Bessel's equations. We see Bessel's equations appear in the telegrapher equation after we take the Fourier Transform. In order to find solutions of Bessel's Equations, we use a method of solving ordinary differential equations called the Frobenius Method, as performed by Asmar [8]. The Frobenius Method involves writing the solution of the ordinary differential equation as a power series and looking for solutions of that form.

Bessel's Equations are any differential equation of the form:

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \tag{6.1}$$

where $x > 0$ and $p \geq 0$. This is called a Bessel equation of order $p$. The corresponding solution is a Bessel function of order $p$.

As previously mentioned, we obtain modified Bessel functions when obtaining a solution to the telegrapher equation. Modified Bessel functions are solutions to the modified Bessel equation which is of the form:

$$x^2y'' + xy' + ((ix)^2 - n^2)y = 0 \tag{6.2}$$

which reduces to:

$$x^2y'' + xy' - (x^2 + n^2)y = 0 \tag{6.3}$$

Thus, the modified Bessel equation is simply the Bessel equation after replacing $x$ with $ix$. The process of solving the modified Bessel equation is very similar to that of the Bessel equation.

Before we understand the solution to both the ordinary Bessel equation and the modified Bessel equation, we must understand the gamma function. The Gamma function appear as a means to simplify the solutions to the Bessel and modified Bessel equation.

## 6.2 The Gamma Function

The Gamma function acts on numbers similar to how factorials act on integers. The definition of the gamma function is as follows:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{6.4}$$

We can use this to rewrite the expression $\Gamma(p+1)$ as follows:

$$\begin{aligned} \Gamma(p+1) &= \int_0^\infty t^p e^{-t} dt \\ &= t^p(-e)^{-t}\Big|_0^\infty + \int_0^\infty e^{-t}(p)t^{p-1} dt \end{aligned}$$

Now we note that the first term in the expression evaluates to 0 when taking the limit as $t \to \infty$ and the second term is equivalent to $p\Gamma(p)$. Thus we obtain the following fact:

$$\Gamma(p+1) = p\Gamma(p). \tag{6.5}$$

We can apply this to integer values of $p$ in order to see the similarity of the Gamma function and the factorial operator. For example for $p = 1, 2, 3$ we get the following:

$$\begin{aligned} &\Gamma(2) = 1\Gamma(1) = 1! \\ &\Gamma(3) = 2\Gamma(2) = 2(1) = 2! \\ &\Gamma(4) = 3\Gamma(3) = 3(2!) = 3! \end{aligned}$$

We can continue this pattern and it follows that

$$\Gamma(p+1) = p! \tag{6.6}$$

for integer values of $p$.

As seen here, we can express the Gamma function of argument $p$ in terms of the Gamma function with argument less than $p$. This makes the Gamma function a useful tool when simplifying recursive relationships.

For non-integer values of $p$, the values of the Gamma function are not as simple. One important case is $\Gamma(1/2) = \sqrt{\pi}$. We can prove that using the following method:
First, we let $t = u^2$ in (6.4). This means that $t^{x-1}dt = 2u^{2x-1}du$ and thus (6.4) becomes:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt = 2\int_0^\infty u^{2x-1}e^{-u^2}du \tag{6.7}$$

Thus, if we look at the case of $\Gamma(1/2)$ in our new definition of the gamma function, we obtain the following result:

$$\Gamma(1/2) = 2\int_0^\infty e^{-u^2}du$$

We know that $e^{-u^2}$ is a Gaussian curve and that the integral of this function from 0 to $\infty$ is equal to $\sqrt{\pi}/2$. Thus we simplify the expression to get:

$$\Gamma(1/2) = \sqrt{\pi} \tag{6.8}$$

Now we can use a combination of (6.5), (6.6), and/or (6.8) to find the value of Gamma functions that have arguments that are integer valued multiples of (1/2). For example:

$$\Gamma(3/2) = \Gamma((1/2)+1) = (1/2)\Gamma(1/2) = \sqrt{\pi}/2$$

Other integer arguments of the gamma function can be found by performing different substitutions for $t$ in (6.4), but for our problem it is sufficient to just mention this case. Now we will apply these facts when solving the ordinary and modified Bessel equation of order $p$.

## 6.3 Solving the Bessel's Equation of order $p$

As previously mentioned, we will follow the use of the Frobenius method as used by Asmar in order to solve the Bessel Equation [8]. So we will rewrite $y$ as a power series. This means our $y$ values will be of the form:

$$y = \sum_{m=0} a_m x^{r+m}, \tag{6.9}$$

where $a_0 \neq 0$. Thus the first and second derivatives of $y$ can be expressed as follows:

$$y' = \sum_{m=0}^{\infty} a_m (r+m) x^{r+m-1} \tag{6.10}$$

$$y'' = \sum_{m=0}^{\infty} a_m (r+m)(r+m-1) x^{r+m-2}. \tag{6.11}$$

Now, we can substitute theses values into the Bessel equation of order $p$ and we obtain:

$$x^2 \sum_{m=0} a_m (r+m)(r+m-1) x^{r+m-2} + x \sum_{m=0} a_m (r+m) x^{r+m-1} \tag{6.12}$$
$$+x^2 \sum_{m=0} a_m x^{r+m} - p^2 \sum_{m=0} a_m x^{r+m} = 0.$$

Note that we can rewrite the $x^2 \sum_{m=0} a_m x^{r+m}$ term by multiplying the $x^2$ into the sum and shifting the sum from starting at $m = 0$ to starting at $m = 2$. Thus we obtain:

$$x^2 \sum_{m=0}^{\infty} a_m x^{r+m} = \sum_{m=2}^{\infty} a_{m-2} x^{r+m} \tag{6.13}$$

Next we will write out the $m = 0$ and $m = 1$ terms of each of the sums. Here we obtain:

$$\Big(a_0(r)(r-1) + a_1(r+1)(r)x^{r+1}\Big) + \Big(a_0(r)x^r + a_1(r+1)x^{r+1}\Big) - \tag{6.14}$$
$$\Big(p^2 a_0 x^r + p^2 a_1 x^{r+1}\Big) + \sum_{m=2} a_{m-2} x^{r+m} + \sum_{m=2} a_m (r+m) x^{r+m-1} +$$
$$\sum_{m=2} a_m (r+m)(r+m-1) x^{r+m-2} = 0$$

Now we will factor out each power of $x$. After this, we rewrite (6.14) as follows:

$$\Big(a_0(r)(r-1(+a_0(r)-p^2a_0\Big)x^r + \Big(a_1(r+1)(r)+a_1(r+1)-p^2a_1\Big)x^{r+1} + \sum_{m=2}^{\infty}\Big[(a_m(r+m)(r+m-1)+a_m(r+m)+a_{m-2}-p^2a_m)x^{r+m}\Big]=0 \tag{6.15}$$

After some simplification of the $a_0$ and $a_1$ terms, we obtain the following expression:

$$\Big(r^2-p^2\Big)a_0x^r+\Big((r+1)^2-p^2\Big)a_1x^{r+1}+\sum_{m=2}^{\infty}\Big(((r+m)^2-p^2)a_m+a_{m-2}\Big)x^{r+m}=0 \tag{6.16}$$

Since the equation is equal to zero, this means that each of the coefficients on $x^r$, $x^{r+1}$, and $x^{r+m}$ must be individually equal to zero. Thus:

$$a_0(r^2-p^2)=0 \tag{6.17}$$

$$a_1((r+1)^2-p^2)=0 \tag{6.18}$$

$$a_m((r+m)^2-p^2)+a_{m-2}=0. \tag{6.19}$$

Since $a_0 \neq 0$, we know that:

$$r^2-p^2=0 \rightarrow (r-p)(r+p)=0 \tag{6.20}$$

This is called the indical equation. We can see here that the indical equation has roots of $r=\pm p$.

When we let $r=\pm p$ in (6.19), and then solve for $a_m$, we obtain the following expression:

$$a_m=-\frac{1}{m(2p+m)}a_{m-2} \tag{6.21}$$

This is a two-step recursion relation of the coefficients of the solution. Since this is a two-step recursion, we solve for the even and odd terms separately.

Dealing with the odd terms first, we note that if we let $r=p$ in (6.12), we get:

$$a_1(p^2+2p+1-p)=0 \rightarrow a_1=0 \tag{6.22}$$

Since $a_1=0$, we can use the recursion relation to tell that all odd term coefficients are going to be equal to zero.

To find the value of the even coefficients, we need to first rewrite the relation where $m = 2k$, in order to ensure that each term in the series corresponds to an even index. Thus, the recursion relation becomes:

$$a_{2k} = -\frac{1}{4k(k+p)}a_{2(k-1)}. \tag{6.23}$$

We can write the first three terms of the sequence out as follows:

$$a_{2(1)} = -\frac{1}{4(1+p)}a_0 \tag{6.24}$$

$$a_{2(2)} = -\frac{1}{4(2)(2+p)}a_2 = \frac{1}{(2^4)(2!)(1+p)(2+p)}a_0 \tag{6.25}$$

$$a_{2(3)} = -\frac{1}{4(3)(3+p)}a_4 = -\frac{1}{(2^6)(3!)(1+p)(2+p)(3+p)}a_0 \tag{6.26}$$

Now, we note that we can substitute these terms in for the coefficient terms in (6.14). When we do this, we get the following sum:

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)(1+p)(2+p)...(k+p)} x^{2k+p} \tag{6.27}$$

Now we will choose a specific value of $a_0$. We will use the following standard expression of $a_0$ [8]:

$$a_0 = \frac{1}{2^p\Gamma(p+1)}. \tag{6.28}$$

Choosing this value for $a_0$ allows us to use the well known fact of the gamma function that states $\Gamma(x+1) = x\Gamma(x)$ in order to simplify the rest of the expression of $y$. In order to do this we will first look at the denominator of the expression. We can simplify this as follows:

$$\begin{aligned}\Gamma(p+1)\Big[(p+1)(p+2)(p+3)...(p+k)\Big] &= \Gamma(p+2)\Big[(p+2)(p+3)...(p+k)\Big] \\ &= \Gamma(k+p+1).\end{aligned} \tag{6.29}$$

Thus, we can simplify (6.27) to get an expression for $y$:

$$y = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)\Gamma(k+p+1)} \Big(\frac{x}{2}\Big)^{2k+p}. \tag{6.30}$$

This is the first solution to the Bessel equation of order $p$. We denote this Bessel function as $J_p(x)$. However, we note that that this is the solution when we let the root of the indical equation be $r = p$. There is another solution when we let $r = -p$. This solution of the Bessel equation can be found with the same method and is of the following form:

$$J_{-p} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)\Gamma(k-p+1)} \left(\frac{x}{2}\right)^{2k-p}. \qquad (6.31)$$

When $p$ is not an integer, $J_p$ and $J_{-p}$ are linearly independent. Thus, we can express the fundamental solution of the Bessel equation when $p$ is not a an integer as follows.

$$y = C_1 J_p + C_2 J_{-p}, \qquad (6.32)$$

where $C_1$ and $C_2$ are arbitrary constants.

However, when $p$ is an integer, we must find a solution that is linearly independent of $J_p$ in order to express the fundamental solution. In order to do this, we consider the following function:

$$Y_p(x) = \frac{J_p \cos(p\pi) - J_{-p}(x)}{\sin(p\pi)} \qquad (6.33)$$

This solution is linearly independent of $J_p$, regardless of whether $p$ is an integer or not. $Y_p$ is called a Bessel function of the second kind of order $p$.

Thus we can express the general solution of Bessel's equation of order p as follows:

$$y(x) = C_1 J_p + C_2 Y_p \qquad (6.34)$$

## 6.4 Solution to the Modified Bessel Equation

Solutions to the modified Bessel equation are closely related to those of the ordinary Bessel Equation. As previously stated, the only difference in the Bessel equation and the modified Bessel equation is that we let $x = ix$. Thus, we would expect that the modified Bessel function would yield complex arguments as opposed to real arguments.

We solve the modified Bessel equation using the same method as we did with the ordinary Bessel equation. We start by assuming the solution and its derivatives to the equation are of the form of a power series. This leads us to the following equation:

$$x^2 \sum_{m=0} a_m (r+m)(r+m-1)x^{r+m-2} + x \sum_{m=0} a_m (r+m)x^{r+m-1} \tag{6.35}$$
$$-x^2 \sum_{m=0} a_m x^{r+m} - p^2 \sum_{m=0} a_m x^{r+m} = 0$$

The only difference between (6.35) and (6.12) is that the $x^2 \sum_{m=0} a_m x^{r+m}$ term is being subtracted rather than added. When we continue to solve this using the Frobinius method, we obtain the following expressions for the coefficients of $x^r$, $x^{r+1}$, and $x^{r+m}$ respectively:

$$a_0(r^2 - p^2) = 0 \tag{6.36}$$

$$a_1((r+1)^2 - p^2) = 0 \tag{6.37}$$

$$a_m((r+m)^2 - p^2) - a_{m-2} = 0 \tag{6.38}$$

We note that again, these are the exact same expressions found in (6.17)-(6.19), with the exception of a factor of $-1$ multiplied to the $a_{m-2}$ in (6.19). Thus, the modified Bessel equation has the same indical equation as the ordinary Bessel equation. Thus the roots of the indical equation are $r = \pm p$. When we use this to find the recursion relationship, we get the following result:

$$a_m = \frac{1}{m(2p+m)} a_{m-2}. \tag{6.39}$$

Just like in the ordinary Bessel equation, we see that the odd terms of the recursion are equal to zero. The process for finding the even coefficients is identical to the ordinary Bessel equation. We obtain the following result:

$$y = \sum_{k=0}^{\infty} \frac{1}{(k!)\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p}. \tag{6.40}$$

This is solution is called the modified Bessel function of order $p$ and is denoted $I_p(x)$, where the argument, $x$ is a complex term.

We can relate the modified Bessel function of order $p$ to the ordinary Bessel function with the following expression:

$$I_p(x) = i^p J_p(ix). \tag{6.41}$$

In order to solve the telegrapher equation, we will need to know the evaluation of the following two modified Bessel functions:

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{(k!)^2} \tag{6.42}$$

$$I_1(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+1}}{(k!)^2(k+1)} \tag{6.43}$$

where $z$ is some complex number.

The modified Bessel function has some properties that will be vital in finding a solution to the telegrapher equation. These properties include these properties can be found in work by Asmar [8]:

$$I_0(0) = 1 \tag{6.44}$$

$$I_n(0) = 0 : n \neq 1 \tag{6.45}$$

$$\frac{d}{dz} I_0(z) = I_1(z) \tag{6.46}$$

In the next chapter, we will derive the form of the standard telegrapher equation using a random walk.

# Chapter 7

# Derivation of the Telegrapher Equation

## 7.1 Introduction

In this chapter, we will use a multi-step random walk in order to derive the telegrapher equation. Recall that in Chapter 4, we used a single-step random walk in order to derive the solution of the heat equation. In that derivation, we represented the solution to the heat equation as a distribution of the probability of finding the random walker at some point in space. In this section, we will essentially track the momentum of the random walker and use the resulting distribution in order to derive the telegrapher equation. This derivation of the telegrapher equation is modeled after the derivation technique used by Weiss and Masoliver [1].

We start this derivation by noting that Fick's Law says the amount of flux is proportional to the gradient of the concentration on the surface of the object. This can be represented by this expression:

$$J(x,t) = -D\frac{\partial c(x,t)}{\partial x}, \tag{7.1}$$

where $c(x,t)$ is the concentration at a specific point in space and $D$ is the diffusion coefficient with units of $\frac{m^2}{s}$. The diffusion constant scales the flux of substance through a medium. We also note that the Conservation Law states that the amount of substance leaving a system is equal to the amount of substance entering the object. Thus when we combine Fick's Law and the Conservation Law, we see that:

$$\frac{\partial c}{\partial t} + \frac{\partial J}{\partial x} = 0. \tag{7.2}$$

From (7.1) we can derive:

$$\frac{\partial J}{\partial x} = -Dc_{xx}. \tag{7.3}$$

Using (7.2) and (7.3), we find that:

$$\frac{\partial c}{\partial t} = Dc_{xx}. \tag{7.4}$$

We know that (7.4) is in the form of the diffusion equation and its solution with the initial condition $c(x, 0) = c_0\delta(x - x_0)$ is:

$$c(x,t) = \frac{c_0}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x - x_0)^2}{4Dt}\right) \tag{7.5}$$

The problem with this solution is that is does not obey relativistic laws. The moment that this solution obtains any sort of time dependence, there is a nonzero value for $c$ at all points of time. It is claimed that the telegrapher equation is an equation similar to the heat equation, but its solutions are confined to a bounded interval whose size is dependent on $t$. The telegrapher equation is of the form:

$$c_{tt} + \frac{1}{T}c_t = v^2 c_{xx}, \tag{7.6}$$

where $T$ is a constant value that has units of time and $v$ is the speed of propagation of the substance through a medium. Physically, $T$ can be thought of as the period of the wave propagating through the medium. The telegrapher equation has properties of both the wave equation and the heat equation. When $T$ approaches $\infty$ , the telegrapher equation behaves similar to the wave equation and when $T \to 0$ and $v \to \infty$ the equation behaves like the heat equation. This tells us that the equation has both particle and wave characteristics. Now we will use the random walk in order to derive this equation.

## 7.2 Random Walk Derivation of the Telegrapher Equation

Consider a random walk on a lattice in which the particle is at $x = j$ and will move to $j \pm 1$ after a single time step. When deriving the heat equation we use the probability of the particle moving in the left or right direction to obtain the form of the equation. In the telegrapher equation we find the probability of moving in the same direction of the previous step in order to find the form of the solution. We will call this probability the persistence probability and we will denote its value as $\alpha$.

Here is an example of this concept: If on the previous step the particle moved from location $j-2$ to $j-1$, then the persistence probability is the probability of the particle moving from $j-1$ to $j$ on the next step.

Now we can write a pair of recursion relations describing the random walk:

- Let $a_n(j)$ be the probability that the random walker is at $j$ at step $n$, while having came from $j-1$ on the previous step.
- Let $b_n(j)$ be the probability that the random walker is at $j$ at step $n$ while having came from position $j+1$.

We now note that $\beta = 1-\alpha$, is the probability of the the particle changing direction on a given time step. We can now write the pair of recursion relations:

$$a_{n+1}(j) = \alpha a_n(j-1) + \beta b_n(j-1) \tag{7.7}$$

$$b_{n+1}(j) = \alpha b_n(j+1) + \beta a_n(j+1) \tag{7.8}$$

Here $\alpha$ and $\beta$ can be equated to a momentum variable as they have a memory of where the particle has been, just as momentum does.

In order to find $a_n(j)$ and $b_n(j)$ we need to find the initial conditions $a_n(0)$ and $b_n(0)$. Also, in order to convert these recursion relations into PDEs we need to scale time, space, and the persistence probability. We can say:

$$x = j\Delta x$$

$$t = n\Delta t$$

where $\Delta x, \Delta t \to 0$.
We have two scaling assumptions that help us to convert (7.7) and (7.8) into PDEs:

$$\lim_{\Delta x, \Delta t \to 0} \frac{\Delta x}{\Delta t} = v \tag{7.9}$$

$$\alpha = 1 - \frac{\Delta t}{2T} \qquad \beta = \frac{\Delta t}{2T} \tag{7.10}$$

where T has dimensions of time.

When we implement these scaling factors we get these PDEs, which are in terms of the continuous forms of the variables $a_n$ and $b_n$, which are denoted as $a$ and $b$:

$$\frac{\partial a}{\partial t} = -v\frac{\partial a}{\partial x} + \frac{1}{2T}(b-a) \tag{7.11}$$

$$\frac{\partial b}{\partial t} = v\frac{\partial b}{\partial x} + \frac{1}{2T}(b-a). \tag{7.12}$$

Now we will write $b$ in terms of $a$ in (7.11) and substitute this into (7.12). When we do this, we obtain the telegrapher equation established in Chapter One:

$$a_{tt} + \frac{1}{T}a_t - v^2 a_{xx} = 0 \tag{7.13}$$

Thus, we can derive the telegrapher equation using similar means as we did the solution of the heat equation. The major difference is that we keep track of where the random walker was located in previous steps, rather than only being concerned with its final position. Now that we have derived the telegrapher equation using a multi-step random walk, we will now solve this equation in the next chapter.

# Chapter 8

# Details on Solving the Telegrapher Equation

## 8.1 Introduction

In this section, we will use a combination of the techniques used in the previous chapters in order to solve and analyze the solution of the telegrapher equation. The process of obtaining this solution is motivated by work done by Masoliver and Weiss [1] [9]. Note from the previous chapter that the telegrapher equation is of the form:

$$U_{tt} + \frac{1}{T}U_t - v^2 U_{xx} = 0, \tag{8.1}$$

where $U$ is a function of $t$ and $x$ and $T$ and $v$ are constants relating to the movement of information through a medium. We can immediately see that the telegrapher equation has characteristics of both the standard heat and wave equations. We would expect the solutions of this equation to have aspects of both the heat and wave equations. We can see this more clearly when looking at the classification of the telegrapher equation.

## 8.2 Classification Theorem in Relation to the Telegrapher Equation

The Classification Theorem of linear, second order partial differential equations is as follows:

Consider the linear second order differential equation of the form:

$$au_{tt} + bu_{tx} + cu_{xx} + du_t + eu_x + fu = 0 \tag{8.2}$$

If $b^2 - 4ac > 0$, then the equation is classified as hyperbolic.
If $b^2 - 4ac = 0$, then the equation is classified as parabolic.
If $b^2 - 4ac < 0$, then the equation is classified as elliptic.

Further details can be found in [4]. The significance of these classifications comes from their canonical forms. The solutions of each type of equation share common characteristics of their behavior. This is because each equation can be reduced to its corresponding canonical form by using a change of variables. The canonical forms are as follows:

$$\begin{aligned} &\text{Hyperbolic: } U_{tt} - U_{xx} + l.o.t... = 0 \\ &\text{Parabolic: } U_{tt} + l.o.t... = 0 \\ &\text{Elliptic: } U_{tt} + U_{xx} + l.o.t... = 0 \end{aligned} \tag{8.3}$$

This reduction of each equation into its canonical indicates that each of the classes of equation will have solutions with similar characteristics.

We can see when comparing the standard heat equation:

$$u_t - ku_{xx} = 0 \tag{8.4}$$

with the form given in (8.2), that $a = 0$, $b = 0$, $c = -k$, $d = 0$. Thus:

$$(0)^2 - 4(0)(-k) = 0, \tag{8.5}$$

which means that the heat equation is parabolic.
Solutions to parabolic hyperbolic differential equations exhibit infinite speed of propagation. For example, the solution of the heat equation on the infinite line has a nonzero value at all points on the line once $t > 0$.

Now we if we look at the standard wave equation;

$$U_{tt} - U_{xx} = 0, \tag{8.6}$$

and compare to (8.2) we see that $a = 1$, $b = 0$, and $c = -1$. Thus;

$$(0)^2 - 4(1)(-1) = 4 > 0, \tag{8.7}$$

which means that the solutions of the wave equation is hyperbolic.
Solutions to hyperbolic differential equations exhibit finite speed of propagation. Solutions are only defined in the region that is based off of the temporal value of the solution.

When looking at the telegrapher equation, we can see that it is of the form of the standard heat equation with a term that is second derivative with respect to $t$ added. When we compare the telegrapher equation with (1), we see $a = 1$, $b = 0$, $c = -v^2$, and $d = \frac{1}{T}$. Thus:

$$(0)^2 - 4(1)(-v^2) = 4v^2 > 0 \tag{8.8}$$

Thus solutions to the telegrapher equation are hyperbolic, just as with the wave equation. This means that the solutions will exhibit finite speed of propagation in the spatial variable with respect to the time variable. However, we would expect the solutions to also have characteristics of the heat equation where the solutions are defined.

## 8.3 Solution of the Telegrapher Equation on the Infinite Line

In order to solve the telegrapher equation, we will perform a Fourier-Laplace transform on the equation. This will give an algebraic expression involving the solution to the telegrapher equation in Fourier-Laplace space. We will then solve the algebraic equation and perform an inverse Fourier-Laplace transform to obtain the solution in $(x, t)$ space.

Note that the Fourier-Laplace transform of the equation is of the form:

$$\int_0^\infty e^{-st} dt \int_{-\infty}^\infty e^{i\omega x}[U_{tt} + \frac{1}{T}U_t - v^2 U_{xx}]dx = 0 \tag{8.9}$$

Before we perform this transform, we make the following change of variables, without loss of generality, in order to simplify the calculation:

$$\tau = \frac{t}{T} \tag{8.10}$$

$$y = \frac{x}{vT} \tag{8.11}$$

After this change of variables, the constants, $v$ and $T$ are essentially set equal to 1. Thus, we obtain the following equation:

$$U_{\tau\tau} + U_\tau - U_{yy} = 0. \tag{8.12}$$

Also, in order to solve the equation we need to establish the following initial conditions:

$$U(0, y) = \delta_0(y) \tag{8.13}$$

$$U_\tau(0, y) = 0 \tag{8.14}$$

Now, we will perform the Fourier transform, with respect to $y$, of both sides of the equation and obtain:

$$\hat{U}_{\tau\tau} + \hat{U}_\tau - i^2\omega^2\hat{U} = 0 \tag{8.15}$$

Thus the equation reduces to:

$$\hat{U}_{\tau\tau} + \hat{U}_\tau + \omega^2\hat{U} = 0 \tag{8.16}$$

where $\hat{\mathrm{U}}$ is a function of $\tau$ and $\omega$. We call $\omega$ the spatial Fourier transform variable.

We can see that the equation is a second order linear ordinary differential equation with respect to $\tau$. We will solve this by now performing a Laplace Transform on both sides of the equation. In order to do this, we need to notice the effects of a Fourier transform on our set of initial conditions:

$$U(0, y) = \delta_0(y) \xrightarrow{\mathcal{F}} \hat{U}(0, \omega) = 1 \tag{8.17}$$

$$U_\tau(0, y) = 0 \xrightarrow{\mathcal{F}} \hat{U}_\tau(0, \omega) = 0. \tag{8.18}$$

Now we can perform the Laplace Transform of (8.16) and obtain:

$$s^2\tilde{U} - s\hat{U}(0, \omega) - \hat{U}_\tau(0, \omega) + \omega^2\tilde{U} + s\tilde{U} - \hat{U}(0, \omega) = 0 \tag{8.19}$$

When we apply the initial conditions in (8.19) and (8.20) and solve for $\tilde{\mathrm{U}}$, we get the solution to the telegrapher equation in Fourier-Laplace space:

$$\tilde{U}(s, \omega) = \frac{s + 1}{s^2 + s + \omega^2}. \tag{8.20}$$

Now we will perform the inverse Fourier-Laplace transform on (8.20) in order to obtain the solution of the telegrapher equation in terms of $\tau$ and $y$. This transform is of the following form:

$$U(\tau, y) = \mathcal{L}^{-1}\Big[\int_{-\infty}^{\infty} e^{i\omega y}\tilde{U}(s,\omega)d\omega\Big]. \tag{8.21}$$

First we will perform the inverse Fourier transform with respect to $\omega$. Thus we will solve the following integral:

$$\hat{U}(s,y) = \int_{-\infty}^{\infty} \frac{s+1}{s^2+s+\omega^2} e^{i\omega y} d\omega \tag{8.22}$$

We will use the Residue Theory in order to solve this integral. We note that the singularities of this expressions occur at:

$$s^2 + s + \omega^2 = 0 \rightarrow \omega = \pm i\sqrt{s^2+s} = \pm a \tag{8.23}$$

Thus, we can rewrite (8.22) as:

$$\hat{U}(s,y) = \int_{-\infty}^{\infty} \frac{s+1}{(\omega-a)} \frac{1}{(\omega+a)} e^{i\omega y} d\omega \tag{8.24}$$

If we simply look at the term $\frac{1}{\omega+a}$ we can rewrite this term as a geometric sum as follows:

$$\frac{1}{(\omega-a)+2a} = \frac{1}{2a}\frac{1}{1-((-1)(\frac{\omega}{2a}))} = \sum_{k=0}^{\infty}(-\frac{\omega}{2a})^k \tag{8.25}$$

We can similarly expand the exponential term and get the following expression:

$$e^{i\omega y} = e^{iy(\omega-a)}e^{iay} = e^{iay}\sum_{k=0}^{\infty}\frac{[iy(\omega-a)]^k}{n!} \tag{8.26}$$

Now, when we expand each of these sums and multiply them by the $\frac{s+1}{\omega-a}$ term, we find the residue of the function:

$$\text{Res}\Big[\frac{s+1}{(\omega-a)}\frac{1}{(\omega+a)}e^{i\omega y}\Big] = -i\frac{s+1}{2\sqrt{s^2+s}}e^{-y\sqrt{s^2+s}} \tag{8.27}$$

Thus we evaluate the Fourier Transform of $\tilde{\mathrm{U}}$ with respect to $\omega$ as:

$$\hat{U}(s,y) = \frac{(s+1/2)+1/2}{2\sqrt{(s+1/2)^2-(1/4)}}e^{-|y|\sqrt{(s+1/2)^2-(1/4)}} \tag{8.28}$$

Now we have to perform an inverse Laplace transform. In order to perform this transform we need to first use the following relationship [7]:

$$\mathcal{L}^{-1}\Big[\frac{e^{-k\sqrt{s^2-a^2}}}{\sqrt{s^2-a^2}}\Big] = I_0(a\sqrt{t^2-k^2})H(t-\mid y\mid) \tag{8.29}$$

We can see where this would be used by rewriting the solution as:

$$\begin{aligned}\hat{U}(s,y) \quad = \quad & [\frac{1/2}{\sqrt{(s+1/2)^2-(1/2)^2}}]e^{-|y|\sqrt{(s+1/2)^2-(1/2)^2}} + \\ & [\frac{s+(1/2)}{\sqrt{2(s+1/2)^2-(1/2)^2}}]e^{-|y|\sqrt{(s+1/2)^2-(1/2)^2}}\end{aligned} \tag{8.30}$$

We can use (8.29) to transform the first term as follows:

$$\begin{aligned}\mathcal{L}^{-1}[[\frac{1/2}{\sqrt{(s+1/2)^2-(1/2)^2}}]e^{-|y|\sqrt{(s+1/2)^2-(1/2)^2}}] = \\ \frac{e^{\frac{-\tau}{2}}}{2}I_0(\frac{\sqrt{\tau^2-y^2}}{2})H(\tau-\mid y\mid)\end{aligned} \tag{8.31}$$

When transforming the second term, we first use the known relationship:

$$\mathcal{L}^{-1}[sF(s)] = f'(t) + \mathcal{L}^{-1}[f(0)] \tag{8.32}$$

where $F(s)$ is the Laplace transform of the function $f(t)$. We can use this relation because the second term is simply the first term with a $s+1/2$ multiplier in the numerator. This means we can use the solution in (8.31) as our $f(t)$. Thus we use this relation to transform the second term as follows:

$$\begin{aligned}\mathcal{L}^{-1}[[\frac{s+(1/2)}{\sqrt{2(s+1/2)^2-(1/2)^2}}]e^{-|y|\sqrt{(s+1/2)^2-(1/2)^2}}] \quad = \\ \frac{d}{d\tau}\Big[\frac{e^{\frac{-\tau}{2}}}{2}I_0\Big(\frac{\sqrt{\tau^2-y^2}}{2}\Big)H(\tau-\mid y\mid)] + \mathcal{L}^{-1}[\frac{1}{2}I_0(i\frac{y}{2}H(-\mid y\mid)\Big],\end{aligned} \tag{8.33}$$

since $H(-\mid y\mid)$ is the Heaviside function centered at 0, and $(-\mid y\mid) < 1$ for all $y$. This means that the $H(-\mid y\mid) = 0$ so the expression simplifies to:

$$= \frac{d}{d\tau}[\frac{e^{\frac{-\tau}{2}}}{2}I_0(\frac{\sqrt{\tau^2-y^2}}{2})H(\tau-\mid y\mid)]. \tag{8.34}$$

In order to find the derivative of this solution, we use the following two derivative facts found in [7]:

$$\frac{d}{dt}I_0(t) = I_1(t) \tag{8.35}$$

and

$$\frac{d}{dt}H(t-x) = \delta(t-x) \tag{8.36}$$

Thus, when we expand this derivative as a three term product rule and simplify, we find that (8.33) simplifies to:

$$\mathcal{L}^{-1}[[\frac{s+(1/2)}{\sqrt{2(s+1/2)^2-(1/2)^2}}]e^{-|y|\sqrt{(s+1/2)^2-(1/2)^2}}] =$$
$$\frac{e^{-\tau\ 2}}{2}\Big[\Big((-1/2)I_0(\lambda)+\frac{\tau I_1(\lambda)}{\lambda}\Big)+\delta(\tau-\mid y\mid)\Big] \tag{8.37}$$

where $\lambda$ is the combination of parameters to simplify the expression:

$$\lambda = \frac{\sqrt{\tau^2-y^2}}{2} \tag{8.38}$$

Thus, we can combine this result with the Laplace transform of the first term and we obtain the solution:

$$U(\tau,y) = \frac{e^{-\tau\ 2}}{4}\Big[\Big(I_0(\lambda)+\frac{2\tau I_1(\lambda)}{\lambda}\Big)+2\delta(\tau-\mid y\mid)\Big] \tag{8.39}$$

Finally, when we replace $\tau$ and $y$ with $t$ and $x$ using the relationships established in (8.10) and (8.11), we obtain the following solution:

$$U(t,x) = \frac{e^{\frac{-t}{2T}}}{2}\Big[\delta(x-vt)+\delta(x+vt)\Big]+\frac{e^{-\frac{-t}{2T}}}{8vT}\Big[I_0(\lambda)+\frac{I_1(\lambda}{2T\lambda}\Big]H(vt-\mid x\mid) \tag{8.40}$$

where $\lambda$ is now:

$$\lambda = \frac{\sqrt{v^2t^2-x^2}}{2vT}. \tag{8.41}$$

Here we have rigorously found a solution to the telegrapher equation. As mentioned at the beginning of this chapter, we know that the solutions of the telegrapher equation are classified as hyperbolic. In the next chapter, we will introduce a bifurcation parameter into the telegrapher equation that causes the solutions of the equation to transition from hyperbolic to parabolic.

# Chapter 9

# Analyzing the Telegrapher Equation with a Bifurcation Parameter

## 9.1 Introduction

In the previous section, we found the solution to the telegrapher equation. In this section we will analyze a potential modification to the telegrapher equation that could more accurately model a relativistic heat equation. This modification is the introduction of a bifurcation parameter to the telegrapher equation. The introduction of this parameter causes the solutions of the equation to transition from hyperbolic to parabolic. This means that the solutions would transition from behaving like the solution of the wave equation, to behaving like the solution to the heat equation. In order to understand how this bifurcation parameter is placed in the telegrapher equation, we will again look at the implications in Classification Theorem.

## 9.2 Classification Theorem as it Relates to the Telegrapher Equation

Consider the telegrapher equation of the form:

$$AU_{tt} + BU_t - CU_{xx} = 0, \tag{9.1}$$

where $A$, $B$, and $C$ are positive constants. According to Classification Theorem, the solutions of this equation would be as follows:
If

$$0^2 - 4(A)(-C) > 0, \tag{9.2}$$

then the solutions are hyperbolic. These hyperbolic solutions will have behavior characteristics similar to the wave equation. In particular they will exhibit finite speed of propagation.
If

$$-4(A)(-C) = 0, \tag{9.3}$$

then the solutions are parabolic. The behaviour of these solutions will resemble that of the heat equation. In particular they will exhibit an infinite speed of propagation.
Lastly, If

$$-4(A)(-C) < 0, \tag{9.4}$$

then the solutions are elliptic.

In the last section, we showed that the telegrapher equation is classified as hyperbolic. However, if we introduce a bifurcation parameter $\lambda$ that is defined as any real number, in the following way:

$$\lambda U_{tt} + U_t - U_{xx} = 0, \tag{9.5}$$

then we can see that the classification of this equation would depend on $\lambda$. We see that the discriminant of (9.5) would be as follows:

$$0^2 + 4(\lambda)(1). \tag{9.6}$$

Thus, whenever $\lambda > 0$, we see that the solutions to the equation are hyperbolic, when $\lambda = 0$, the solutions are parabolic, and when $\lambda < 0$, the solutions are elliptic. Thus, as $\lambda$ transitions from a positive value to a negative value, we would expect that the solutions to the telegrapher equation would transition from behaving like that of the wave equation, to that of the heat equation. Also, we can tell that setting $\lambda$ equal to zero gives us:

$$U_t - U_{xx} = 0, \tag{9.7}$$

which is the exact form of the standard heat equation. In order to understand the effects of this bifurcation parameter on the telegrapher equation, we will solve the telegrapher equation and observe how to behavior of $\lambda$ effects the solutions of the telegrapher equation.

## 9.3 Solving the Telegrapher Equation with a Bifurcation Parameter

Solving this modified version of the telegrapher equation is done in a very similar manner as solving the standard telegrapher equation. We just need to carefully track the $\lambda$ as we perform these transformations. We will use the same initial conditions that were used in solving the standard telegrapher equation. We will also make the same change of variables that we made in order to simplify the calculations when solving the standard telegrapher equation:

$$y = \frac{x}{vT}, \tag{9.8}$$

$$\tau = \frac{t}{T}. \tag{9.9}$$

Thus, when we perform the Fourier transform on the modified telegrapher equation we obtain:

$$\lambda \hat{U}_{\tau\tau} + \hat{U}_{\tau} + \xi^2 \hat{U} = 0 \tag{9.10}$$

In a similar manner, we can perform the Laplace transform on (9.10), and we obtain an expression that is similar to the solution to the telegrapher equation. The solution to this modified telegrapher equation in Fourier-Laplace space is as follows:

$$\tilde{U}(s,\tau) = \frac{\lambda s + 1}{\lambda s^2 + s + \xi^2} \tag{9.11}$$

Now, following the same process as in the previous case, we will invert the Fourier transform in order to obtain the solution in Laplace space. In order to do this, we must perform the following calculation:

$$\int_{-\infty}^{\infty} e^{i\xi y} \tilde{U}(s,\xi) d\xi. \tag{9.12}$$

In order to perform this calculation, we must find the residue of $Ue^{i\xi y}$ at the singularities. We can see that the singularities of this expression occur at:

$$\xi = \pm i\sqrt{\lambda s^2 + s}. \tag{9.13}$$

Calculating the residue of $\tilde{U}e^{i\xi y}$ closely follows the calculation we performed last section. We find that the residue is:

$$\mathrm{Res}(\tilde{U}e^{i\xi y}, i\sqrt{\lambda s^2+s}) = \frac{\lambda s+1}{\sqrt{\lambda s^2+s}}e^{-y\sqrt{\lambda s^2+s}} \tag{9.14}$$

Thus the inverse Fourier transform of (9.8) is as follows:

$$\hat{U}(s,y) = \frac{\lambda s+1}{\sqrt{\lambda s^2+s}}e^{-y\sqrt{\lambda s^2+s}}. \tag{9.15}$$

Next, we will invert the Laplace transform in order to obtain the solution in real space. Again, this process will be very similar to that of standard telegrapher equation, however we must track the $\lambda$ in order to see where it appears in the solution in real space.
The first step of performing this inverse transform is to rewrite the expression using a few algebraic steps as follows:

$$\hat{U}(s,y) = \sqrt{\lambda}\Big[\frac{s+1/(2\lambda)}{\sqrt{(s+(1/(2\lambda))^2-(1/(2\lambda))^2}} \tag{9.16}$$

$$+\frac{1/(2\lambda)}{\sqrt{(s+(1/(2\lambda))^2-(1/(2\lambda))^2}}\Big]e^{-y\sqrt{s+(1/(2\lambda))^2-(1/(2\lambda))^2}}, \tag{9.17}$$

where $\lambda \neq 0$. Now, we use the same well-known properties of the inverse Laplace transform that we used in the previous section to obtain the solution to the (9.16) and (9.17) of this expression. The inverse Laplace transform of (9.16) is as follows:

$$\frac{e^{-\frac{\tau}{2\lambda}}}{2\sqrt{\lambda}}\Big[\Big(\frac{\tau}{2\lambda\sqrt{\tau^2-y^2}}I_1\Big(\frac{\sqrt{\tau^2-y^2}}{2\lambda}\Big)-\frac{1}{2\lambda}I_0\Big(\frac{\sqrt{\tau^2-y^2}}{2\lambda}\Big)\Big)H(\tau-\mid y\mid)$$
$$+\delta(\tau-\mid y\mid)\Big] \tag{9.18}$$

Also, note that the inverse Laplace transform of (9.17) is as follows:

$$\frac{e^{-\frac{\tau}{2\lambda}}}{2\sqrt{\lambda}}I_0\Big(\frac{\sqrt{\tau^2-y^2}}{2\lambda}\Big)H(\tau-\mid y\mid) \tag{9.19}$$

When we combine the expressions in (9.18) and (9.19), and simplify the result, we obtain the following solution for the solution to the modified telegrapher equation in real space:

$$U(\tau, y) = \frac{e^{-\frac{\tau}{2\lambda}}}{4\lambda}\Big[\big[(2\sqrt{\lambda}-1)I_0\Big(\frac{\sqrt{\tau^2-y^2}}{2\lambda}\Big) + \Big(\frac{\tau}{\sqrt{\lambda\tau^2-\lambda y^2}}I_1\Big(\frac{\sqrt{\tau^2-y^2}}{2\lambda}\Big)\big]H(\tau- \mid y \mid) + 2\sqrt{\lambda}\delta(\tau- \mid y \mid)\Big] \tag{9.20}$$

We can see that this is a very similar solution to the one we found for the standard telegrapher equation. Now we will analyze some of the implications that this bifurcation parameter has on the solution to the telegrapher equation.

### 9.3.1 Analyzing the Solution to the Telegrapher Equation with a Bifurcation Parameter

First, as previously mentioned, when $\lambda = 0$ we obtain the heat equation. Though, we have established that $\lambda \neq 0$ in (9.20) we can analyze the implications of setting $\lambda = 0$ in Fourier, Laplace, and Fourier-Laplace space found in (9.10), (9.15), and (9.11) respectively. This is valid because these solutions had no restriction on the value of $\lambda$. First, we will analyze the effects of setting $\lambda = 0$ in Fourier space.

### 9.3.2 Choosing $\lambda = 0$ for the solution in Fourier Space

Notice that when we set $\lambda = 0$ in (9.10), we obtain the following equation:

$$\hat{U}_\tau + \xi^2\hat{U} = 0. \tag{9.21}$$

This is a first order, linear ordinary differential equation with respect to $\tau$. Thus, when we multiply each side by a integrating factor of $\mu = e^{\xi^2\tau}$ to each side of the equation. When we do this, we obtain the following:

$$\frac{d}{d\tau}\Big(e^{\xi^2\tau}\hat{U}\Big) = 0. \tag{9.22}$$

After integrating both sides with respect to $\tau$ we get the following expression:

$$\hat{U}e^{\xi^2\tau} = f(\xi) \tag{9.23}$$

where $f(\xi)$ is an arbitrary function of $\xi$. Thus solving for $\hat{U}$ we obtain the solution to the modified telegrapher equation in Fourier-space:

$$\hat{U} = f(\xi)e^{-\xi^2\tau} \tag{9.24}$$

Now, we will obtain the solution in real space by taking the inverse Fourier transform. This calculation is done with the following calculation:

$$U(\tau, y) = \int_{-\infty}^{\infty} e^{-\xi^2} f(\xi) e^{i\xi y} d\xi \tag{9.25}$$

Using the well known identity that the inverse Fourier transform of a Gaussian is another Gaussian with a normalizing factor [7], and our initial conditions, we obtain the following result:

$$U(t, x) = \frac{1}{\sqrt{4\pi\tau}} e^{\frac{-y^2}{4\tau}}, \tag{9.26}$$

which is the same result that we have obtained in previous sections. This agrees with our expectations that setting $\lambda = 0$ will yield the solution to the heat equation. Now we will observe the effects of letting $\lambda = 0$ for the solution in Laplace space.

### 9.3.3 Choosing $\lambda = 0$ for the solution in Laplace Space

Note that when we set $\lambda = 0$ in the solution to the modified telegrapher equation in Laplace space found in (9.15), we get:

$$\hat{U}(s, y) = \frac{1}{\sqrt{s}} e^{-y\sqrt{s}} \tag{9.27}$$

Now we will perform the inverse Laplace transform on (9.27) in order to obtain the solution in real space. In order to perform this inverse transform, we will use the following well-known property of the inverse Laplace transform:

$$\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}} e^{-k\sqrt{s}}\right] = \frac{1}{\sqrt{4\pi t}} e^{\frac{-k^2}{4t}} \tag{9.28}$$

Thus, the solution to the modified telegrapher equation in real space when $\lambda = 0$ in Laplace space is as follows:

$$U(\tau, y) = \frac{1}{\sqrt{4\pi\tau}} e^{\frac{-y^2}{4\tau}} \tag{9.29}$$

Again, this solution agrees with what we have found in previous sections, along with the solution we obtained when we set $\lambda = 0$ in the solution obtained in Fourier space.

Lastly, we will analyze the behavior of the solution in Fourier-Laplace space when we set $\lambda = 0$

### 9.3.4 Choosing $\lambda = 0$ for the solution in Fourier-Laplace Space

We see that when we let $\lambda = 0$ in the solution to the modified telegrapher equation solution in Fourier-Laplace space found in (9.11), the expression reads:

$$\tilde{U}(s, \xi) = \frac{1}{s + \xi^2} \tag{9.30}$$

Note that this is the same solution that we obtained in the previous section for the heat equation in Fourier-Laplace space. Thus, when we perform the inverse Fourier-Laplace double transform, we will obtain the solution to the heat equation in real space. Thus, the solution is:

$$U(\tau, y) = \frac{1}{\sqrt{4\pi\tau}} e^{\frac{-y^2}{4\tau}} \tag{9.31}$$

which agrees with our previous solutions to the heat equation.

This implies that, while the requirement that $\lambda \neq 0$ holds in the solution to the modified telegrapher equation in real space, we can let $\lambda = 0$ in Fourier, Laplace, or Fourier-Laplace space and see that the solution is identical to that of the heat equation.

# Chapter 10

# Conclusion

## 10.1 Conclusions Concerning the Telegrapher Equation with a Bifurcation Parameter

In this paper, we have derived a solution to the telegrapher equation with a bifurcation parameter that causes a transition in the solutions of the equation from hyperbolic to parabolic. We have shown that the solution to this modified telegrapher equation is that of a heat equation when we choose $\lambda = 0$ in Fourier, Laplace, and Fourier Laplace space yields the solution to the standard heat equation. However, the solution of the telegrapher equation with a bifurcation parameter in real space requires that $\lambda \neq 0$. This solution represents a distribution of solutions to the telegrapher equation as the classification of the solutions transition from hyperbolic to parabolic. This means that we have obtained a solution to the telegrapher equation that behaves just like that of the heat equation when $\lambda = 0$ and behaves similar to the solution of the wave equation when $\lambda > 0$. This is a solution that we believe could more accurately represent that of a relativistic heat equation. Beyond that, this solution gives us a visual representation of the transition of the solutions of the telegrapher equation transitioning from hyperbolic to parabolic and even into the elliptic range. This is an interesting result because these classifications of solutions have very different behaviors. This could potentially give us insight into the behavior of these solutions as they transition between different classifications.

## 10.2 Future Work

Future work related to this subject would include finding some bifurcation parameter that corresponds to a solution in real space without the requirement of $\lambda \neq 0$. This would give us a solution that would continuously transition between different classifications. Beyond this, future work could include analyzing the behavior of the solution as $\lambda$ approaches zero in order to understand the behavior of the telegrapher equation as it's solutions transition between different classifications. This could potentially give us insight into the behavior of other second order, partial differential equations as their solutions transition between different classifications.

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