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Comparison of Non-Prosthetic and Prosthetic Strides in a Pendulum-Based Model

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2 Acknowledgement

I want to thank Dr. Catlla for her guidance during this project. She never let me "fall off the edge" but she did let me get close a few times – and I am better for it. I also want to thank my family for their constant and unwavering support. I could not have done this without all of you!

3 Abstract

In this paper,we explore the differences in a non-prosthetic and prosthetic single stride. We accomplish this by developing a model based on a forced, triple pendulum. We use this model to describe a single stride and alter where the internal force comes from to simulate a prosthetic and non-prosthetic stride. We numerically solve our model with Matlab. We find that our model qualitatively represents the energy gap between a prosthetic and non-prosthetic stride. Our model also agreed qualitatively with alterations of the prosthetic designed to decrease the energy gap between the two strides.

4 Introduction

When I began to consider embarking on a senior thesis, I was excited about the idea of getting to work on a project that combines my two passions: math and biology; for most of my academic career, I have not been able to study this intersection. Biology is perceived as too complex for math, lacking the mathematical foundation of physics, and mathematical equations often become over-determined in order to fully describe biological systems. However, even overly simplified equations, can give us tremendous insight into the complex systems they describe. Human gait is no exception. The act of walking, which may seem like a simple task to the average person, is complex and difficult to model for a mathematician.

Previous papers have attempted to replicate gait with a variation of a pendulum and Newton's second law [7, 3]. However, these papers have not accounted for a prosthetic device, which is what we are interested in.

The World Health Organization estimates that 30 million people are in need of prosthetic and orthotic devices worldwide [4]. What's more, "the number of people with limb loss in the United States could double — from the 1.6 million people estimated in 2005 to 3.6 million in 2050 — largely due to vascular disease often caused by diabetes" [4]. Thus, studies surrounding gait and prosthesis have an impact on a large, global community. Additionally, a prosthetic can have an immense impact on the gait of an individual. These impacts range from knee instability to decreased trunk motion [5]. There is also evidence that prosthetic gait uses more energy than non-prosthetic gait [6]. So, studying the cause of these deviations in an effort to design devices that can combat them, is of great importance.

In this thesis, we develop a model that allows us to compare non-prosthetic and prosthetic strides. A stride is a single step in a gait. Based on our model, we will alter the prosthetic to consider the impact of changes on a prosthetic ease gait. In Section 1, we give a background of what gait is in the biomechanical sense, and we explore what mathematical models that have been used in the field to describe and study gait [8, 3, 7]. In Section 2, we dive into the development of our model, including assumptions and initial conditions. Once we have an understanding of the model, we move to explore numerical results in Section 3. We see that the model qualitatively describes both a prosthetic and non-prosthetic stride and illustrates the energy increase needed in a prosthetic stride.

5 Background

To understand my thesis, we must have a basic understanding of the two main categories that drive this project: biomechanics and mathematical modeling.

5.1 Biomechanics

Before we can explore the ways in which gait varies, we first need the proper terminology. Some commonly used terms to describe the phases of gait are given in Tables 1 and 2. [5] These terms are used to describe all forms of human gait. If there are deviations between individuals, they will be described using these terms. Additionally, to describe prosthetic and non-prosthetic gait, we rely on these terms so we can have a basis for our discussion.

Table 1: Stance (Standing) Phase

Heel Strike	The point at which the heel touches the ground
Loading Response	After heelstrike the weight is shifted onto the stance leg
Midstance	The point at which the tibia is perpendicular to the ground
Terminal Stance	This is the phase moving into hip extension until just before preswing
Pre-swing	Knee flexion and and toe-off just before hip flexion occurs

Table 2: Swing Phase

Initial Swing	Hip begins flexion, knee stays in flexion
Mid Swing	Hip flexion moves swing leg past stance leg as it prepares for knee extension
Terminal Swing	Maximum hip flexion paired with full knee extension as the body shifts forward in preparation for the heel strike

When comparing the gait of a non-amputee to the gait of one with a prosthetic, there are both obvious and more slight differences. One obvious difference is that someone with a poorly fitted prosthetic may have a noticeable limp through the midstance where a non-amputee does not [5]. An example of a more slight difference is that in an effort to smoothen their gait, amputees often use different muscle groups compared to someone with a nonprosthetic gait, which may not be obvious to an observer. Amputee gait is categorized on where the amputation was done and there are certain characteristics that belong to these categories. This terminology describing the types of prosthetic gait and these characteristics are summarized in Table 3 [5]. With both transtibial and transfemoral gaits, we see an increase in energy usage compared to a non-prosthetic gait.

We define each type of amputation in two categories: traumatic or vascular [5]. Traumatic is due to

Table 3: Types of Gait with a Prosthetic

Transtibial Gait	This is an individual with a below the knee amputation and reduced ankle movement.
Transtibial Gait Characteristics	There is a decrease in the range of extension at the hip joint (about one-half of the opposite limb), the stance time on the nonprosthetic leg is greater than the stance time on the prosthetic, and finally we see an increase in the heel strike time.
Transfemoral Gait	This is an individual with an above the knee amputation and it is focused on preventing the knee from buckling.
Transfemoral Gait Characteristics	This gait is 30 percent slower than non prosthetic gait, there is more asymmetry, the sound leg will have an increased ground reaction force and more hip range with single limb stance than the prosthetic side.

injury and vascular is due to a vascular disease of sorts. In a traumatic transtibial gait we see a 25 percent energy increase and a vascular transtibial will have a 40 percent energy increase. A traumatic transfemoral gait will show a 68 percent energy increase and a vascular transfemoral will show a 100 percent energy increase[6]. We are now equipped with the proper information to advance into what models of gait look like.

5.2 Modeling

Most models of gait consider the limbs as pendula and use Newton's Second Law (F=ma) or balance around angular momentum[7, 8]. In this section we will present the models that inspired our model.

We will start our modeling discussion at the same point I started my research with a paper called: "The Simplest Walking Model" [7]. This model viewed gait as the movement of an inverted simple pendulum couple to a pendulum whose support moves through an arc. This model is governed by the following two dimensionless equations:

$$\ddot{\theta(t)} - \sin(\theta(t) - \gamma) = 0 \tag{1}$$

$$\ddot{\theta(t)} - \ddot{\phi(t)} + \ddot{\theta(t)}^2 \sin(\phi(t)) - \cos(\theta(t) - \gamma)\sin(\phi(t)) = 0, \tag{2}$$

where γ is the slope of the ramp and θ and ϕ are shown in Fig. 3.

To understand the format of equations (1) and (2), lets review the equations of a double pend. These equations are based on the equations that govern a double pendulum and are as follows [9]:

$$(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^{\,2} \sin(\theta_1 - \theta_2) + g m_1 m_2 \sin(\theta_1) = 0 \tag{3}$$

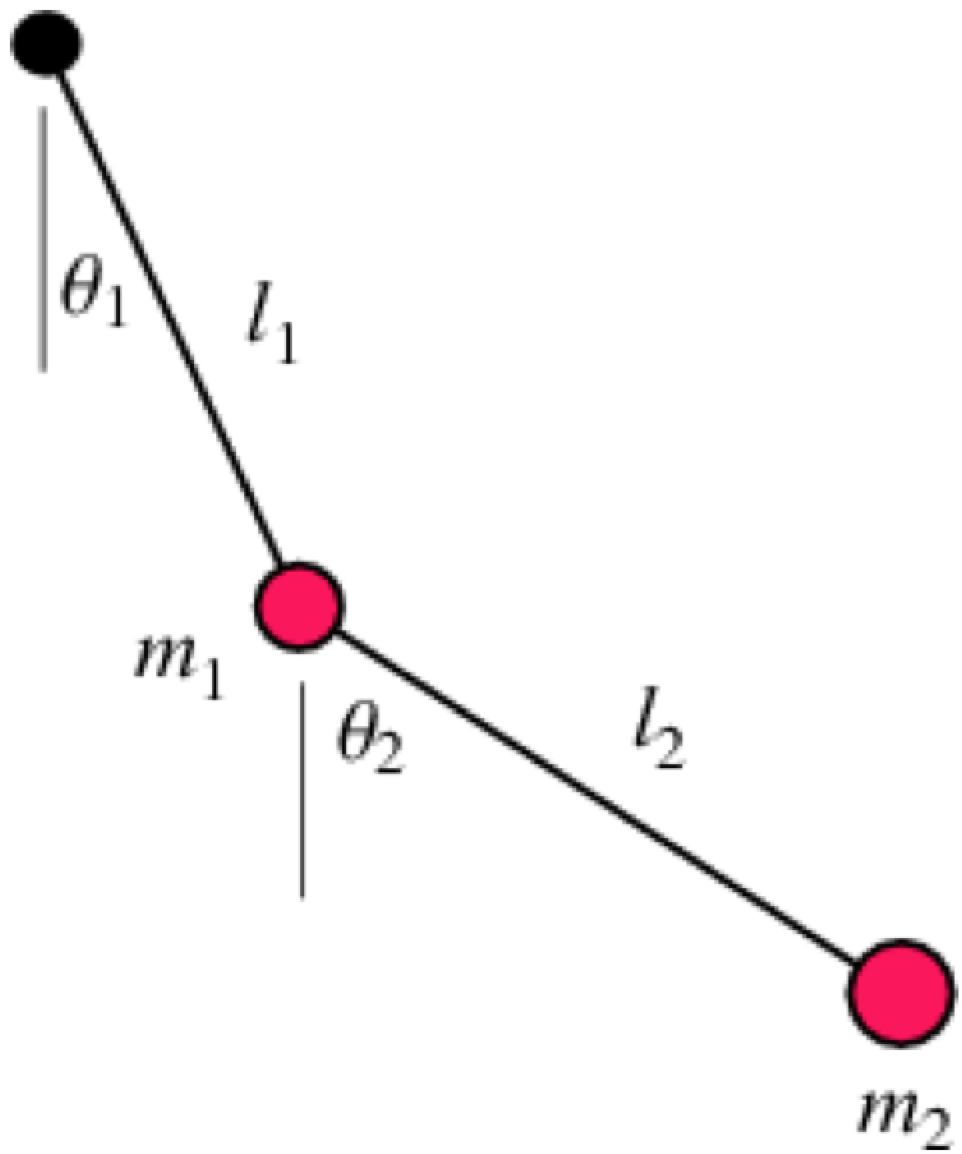


Figure 1: A labeled illustration of a double pendulum [10]

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta_1}^2 \sin(\theta_1 - \theta_2) + g m_2 \sin(\theta_2) = 0 \tag{4}$$

where θ_1, θ_2, l_1, l_2, m_1, and m_2 are shown in Fig. 1 and g represents gravity. These equations are derived by balancing angular momentum. Equation (2), looks at the legs as a double pendulum whose support (the hip) moves through an arc and represents the angular momentum balance about the hip. You can think of this as if m_1 from Fig. 1 is the hip. The simplest walking model bases the stance leg on an inverted pendulum. The equation for an inverted pendulum is:

$$\ddot{\theta} - \frac{g}{l} \sin\theta = 0 \tag{5}$$

where θ and l are shown in Fig. 2 and g represents gravity. An inverted pendulum has the same equation as a simple pendulum, except they differ by a sign. This is because their stability is flipped as a simple pendulum is stable at $\theta = 0$. Note the similarity between equation (5) and equation (1).

This was a great place to start my research as it was simple and had open source Matlab code that I could work with (https://www.mathworks.com/matlabcentral/fileexchange/56859-passive-dynamic-walking). This paper's authors used numerical solvers to show that shorter gait is less stable but more efficient than longer gait. Shorter gait is described as the change in θ from the heel strike to the terminal swing is shorter. However, they do make sure to note that this would not necessarily be true if the model had non-negligible foot masses [7].

The simplest walking model was not a viable model for my end goal. Not only was the model too simple, with no knee joint and few parameters for me to alter, but the code to solve the model numerically was finicky. When I tried to make alterations to the code, it essentially "freaked out" giving us a wildly unrealistic result where it behaved like a chaotic double pendulum.

The next model we will consider is of a self-impacting double pendulum over a single stride [8]. The authors used the Lagrangian (kinetic energy minus potential energy) to derive the following equations:

$$\frac{(m_1 + 3m_2)}{3} l_1^2 \ddot{\theta}_1 + \frac{m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)}{2} + \frac{m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)}{2} + \frac{(m_1 + 2m_2)}{2} g l_1 \sin(\theta_1) = 0 \tag{6}$$

and

$$\frac{m_2 l_2^2 \ddot{\theta}_2}{3} + \frac{m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2)}{2} - \frac{m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)}{2} + \frac{m_2 g l_2 \sin(\theta_2)}{2} = 0 \tag{7}$$

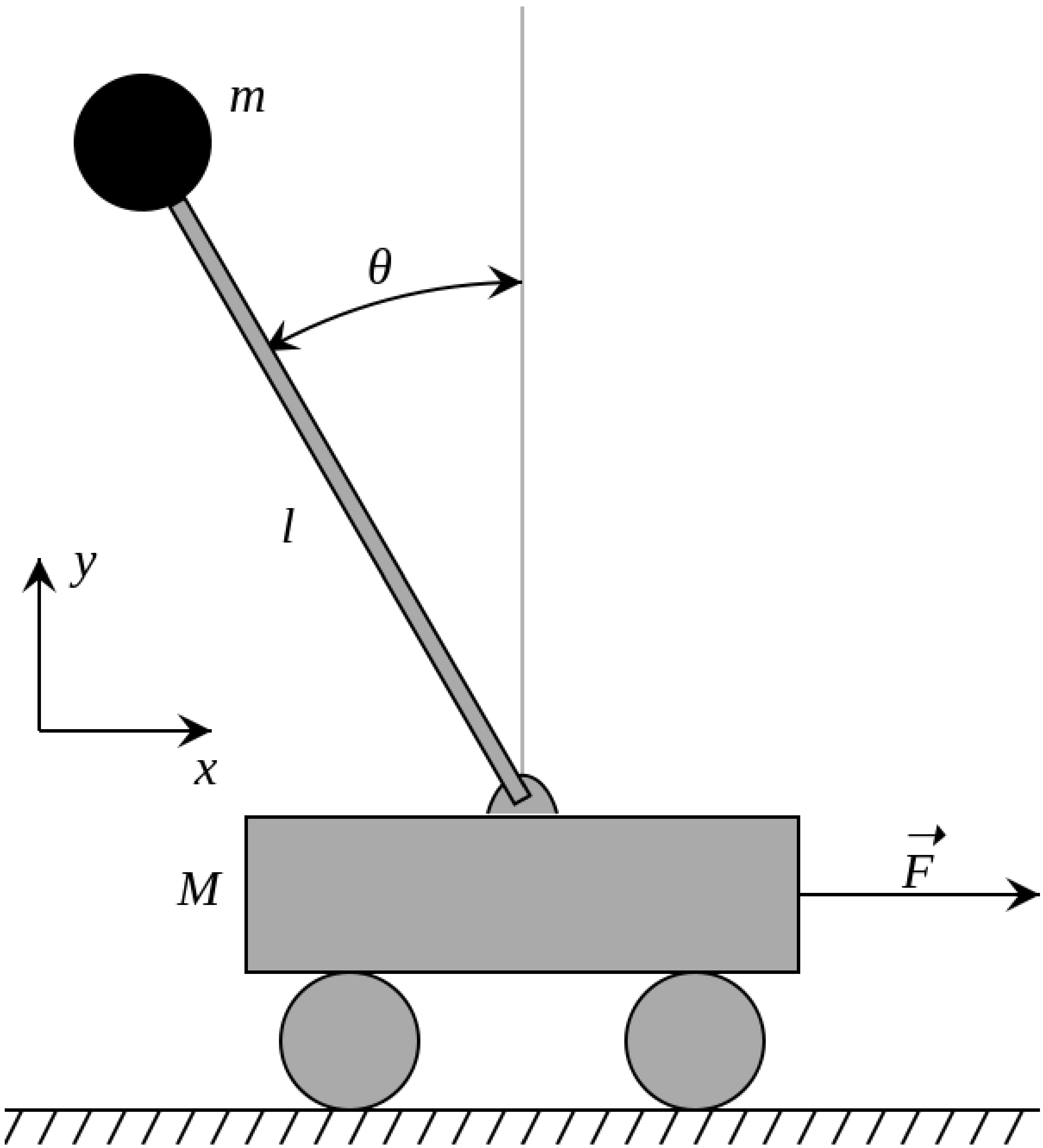


Figure 2: A labeled graphic of an inverted pendulum [2]

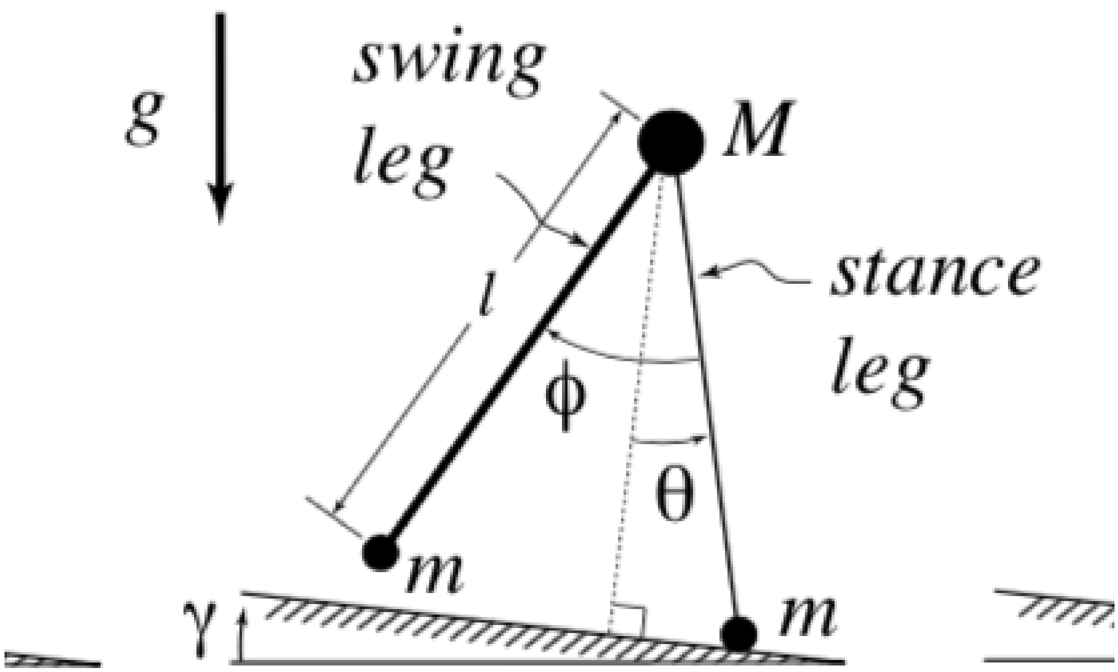


Figure 3: Illustration for the"Simplest Walking Model" equations [7]

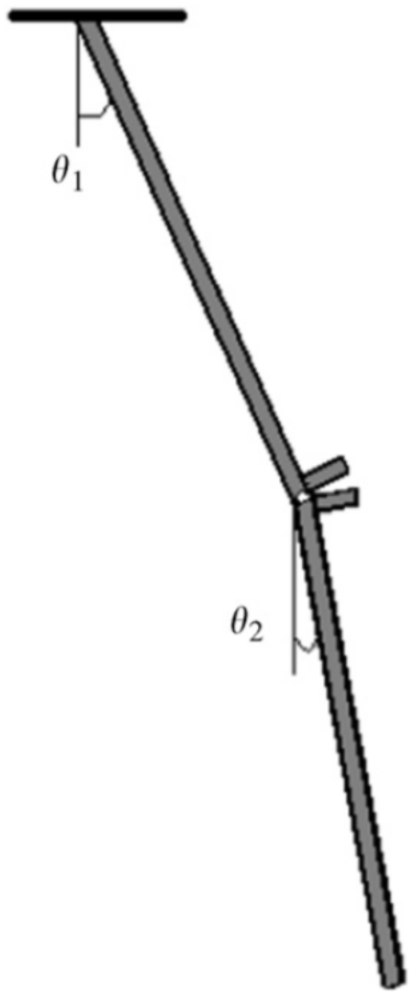


Figure 4: Image of a self-impacting double pendulum [8]

where m_1 is the mass of the upper link, m_2 is the mass of the lower link, l_1 is the length of the upper link, l_2 is the length of the lower link, and θ_1 and θ_2 are as shown in Fig. 4.

Sanghi etal used these equations not to study gait, but to analyze a self-impacting double pendulum [8]. They considered multiple periods, and they found that the energy decreased after each cycle of the self-impacting double pendulum[8]. A subsequent paper, found that a self-impacting double pendulum was an excellent representation of a single leg as it operated considerably well in various conditions of human walking[3].

Once I had a good understanding of how they created the self-impacting double pendulum model, I worked on writing code to numerically solve these equations. However, these equations were difficult to solve using Matlab. They were second order ODEs that could not be transformed into a system of first order ODEs, so I could not use ODE45. In an effort to obtain any sort of numerical result, we turned to the appendix of the paper. Here the authors used some algebraic tricks and specific conditions to turn the equation into :

$$I\ddot{\phi} + c\dot{\phi} + k\phi = 0, \tag{8}$$

where

$$I = \frac{(4r_m + 3)m_2 l_1^2}{6(6r_l^2 + 6r_l + 2r_m r_l^2 + 2)} \tag{9}$$

and

$$r_m = \frac{m_1}{m_2} r_l = \frac{l_1}{l_2} \tag{10}$$

This could be solved numerically, but it assumed we were only looking at the heel strike, which made it inappropriate for my question on prosthesis. This again, was not exactly what I was looking for, but I now had a strong understanding of what the general model looked like, and I was ready to begin constructing my own model.

6 Model Development

Based on the previously described models, I settled on modeling a single stride, as the information I cared about – how a prosthetic affects walking – was present in a single stride. So, I wanted to construct a model of a stride as a linked, forced pendulum. We chose a forced triple pendulum with angles laid out in Fig. 5.

From Fig. 5, we can visualize what each angle does throughout the course of the stride. Following the convention that clockwise is negative and counterclockwise is positive, we expect θ_1 to start open in pre-swing at a 45 degree angle and pass zero during the mid-swing before increasing again. For θ_2, we expect to start with a large magnitude in the pre-swing and approach and pass through zero through the midswing, before increasing again. Finally, we expect θ_3 to start open, but with the opposite sign from θ_1 and θ_2 and approach zero. From there we decided on a forced pendulum where the internal force is the effort put in by the body to propel through a stride. We can also think of this as where the push is coming from to take a step.

Once we had that settled, we moved to writing our equations, based on the model of a simple pendulum, where F=ma is using to derive the equation. The derivation is as follows:

$$
\begin{aligned}
F &= ma && (11)\\
mg\sin(\theta) &= ml\ddot{\theta} && (12)\\
\ddot{\theta} + \omega^2 sin\theta &= 0 && (13)
\end{aligned}
$$

where θ is shown in Fig. 6 and ω is the square root of g/l where g is gravity and l is shown in Fig. 6.

In order to create our model, we followed the same steps. The equation governing θ_1 is derived by the following steps:

$$
\begin{aligned}
F &= ma && (14)\\
mg\sin(\theta_1) - f_{mus1}(t) &= ml\ddot{\theta_1} && (15)\\
\ddot{\theta_1} &= \frac{g}{l_1}\sin(\theta_1) - \frac{f_{mus1}(t)}{m_1 l_1} && (16)\\
\ddot{\theta_1} + \omega^2 sin(\theta_1) - f_{int1}(t) &= 0 && (17)
\end{aligned}
$$

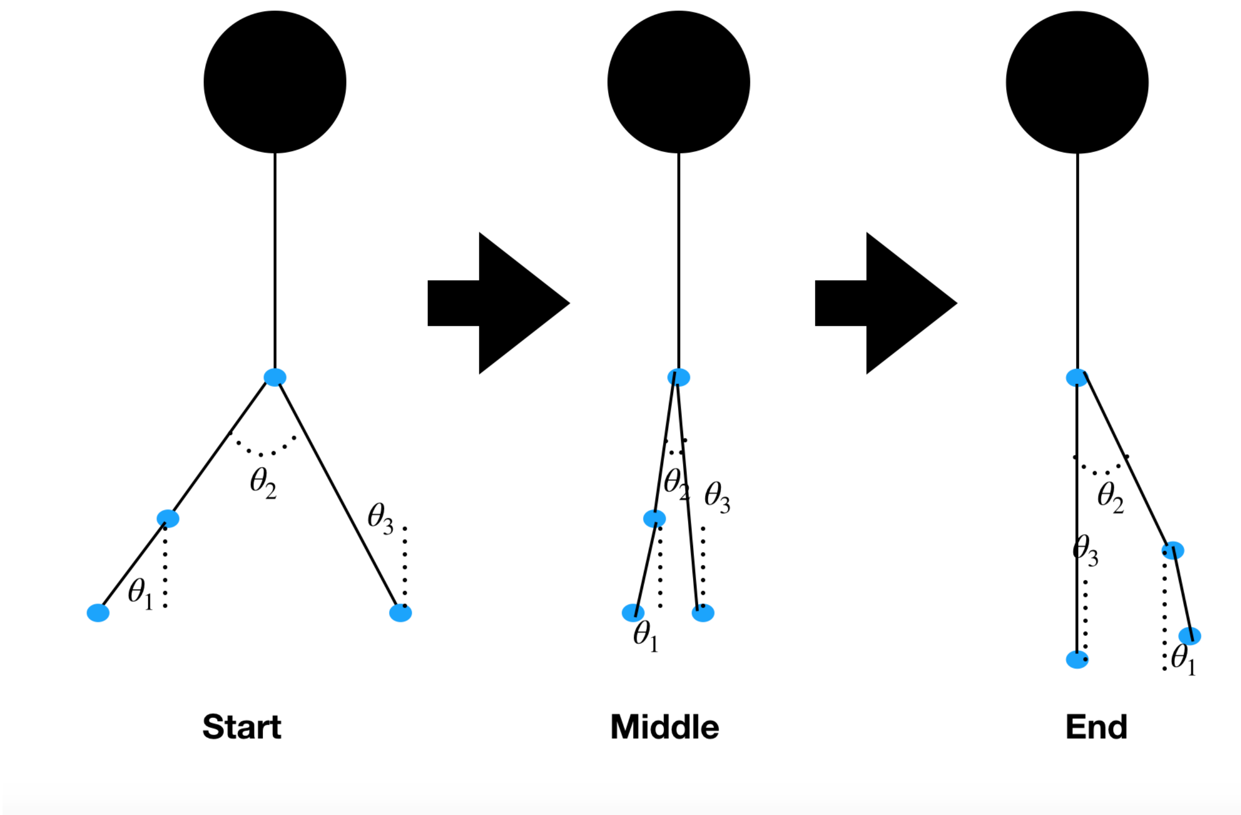


Figure 5: Reference for the angles in our model

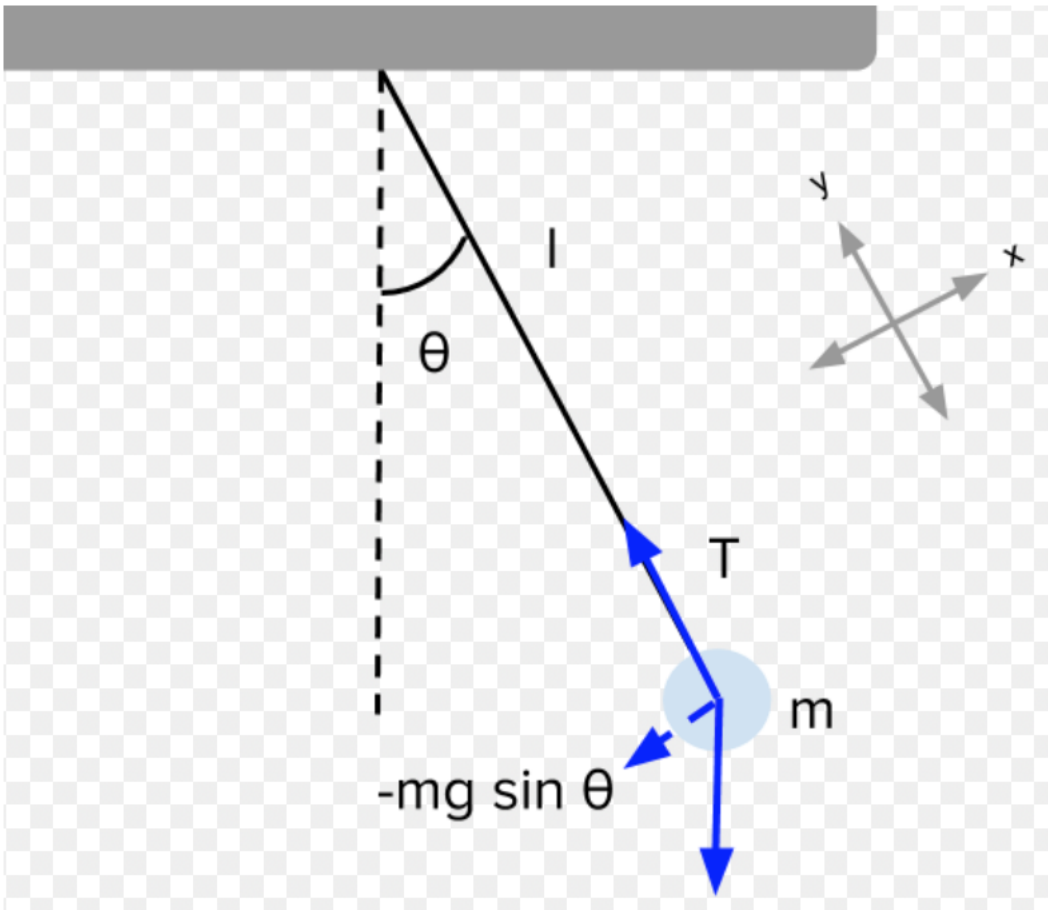


Figure 6: Labeled illustration of a simple pendulum [1]

where θ_1 is shown in Fig. 5, ω is the square root of g/l_1 where g is gravity, l_1 is the length of the segment as shown in Fig. 5 and $f_{int1}(t)$ is $\frac{f_{mus1}(t)}{m \cdot l}$. This leaves us with a dimensionless equation for θ_1.

The derivations of the θ_2 and θ_3 equations are similar, except we also take into account the effect the lower pendulum segments have on the upper segments. The derivation of the θ_2 equation is as follows:

$$\begin{aligned} F &= ma && (18)\\ mg\sin(\theta_2) - f_{mus2}(t) - Acos(\theta_1) &= ml\ddot{\theta_2} && (19)\\ \ddot{\theta_2} &= \frac{g}{l_2}\sin(\theta_2) - \frac{f_{mus2}(t)}{m_2 l_2} - \frac{A}{m_2 l_2} && (20)\\ \ddot{\theta_2} + \omega^2 sin(\theta_2) - f_{coup}(\theta_1) - f_{int2}(t) &= 0 && (21) \end{aligned}$$

where each θ is shown in Fig. 5, ω is the square root of g/l_2 where g is gravity and l_2 is the length of the segment as shown in Fig. 5 and $f_{coup}(\theta_1)$=(A/$m_2 l_2$)cos(θ_1). $f(\theta)$ is the coupling term. We choose an A such that the coefficient on f_{coup} is 1. Note, in the $\ddot{\theta_2}$ equation, f(θ) is reliant on θ_1; this is how the θ_1 affects θ_2. This leaves us with a dimensionless equation for θ_2. And the same process gives us the θ_3 equation:

$$\ddot{\theta_3} + \omega^2 sin(\theta_3) - f_{coup}(\theta_2) = 0 \qquad (22)$$

where each θ is shown in Fig. 5, ω is the square root of g/l_3 where g is gravity and l_3 is the length of the segment as shown in Fig. 5 and $f_{coup}(\theta_2)$=(A/$m_3 l_3$)cos(θ_2). Note, l_1, l_2 and l_3 are not absolute length values, rather ratios between the lengths of the parts of the leg where l_1=.4, l_2=.6 and l_3=1 and similarly for m_1,m_2,m_3. This leaves us with a dimensionless equation for θ_3. Our complete model is equations (17),(21),(22).

The coupling term, $f_{coup}(\theta)$, is how we connect the segments of the pendulum. We chose the form of cos since we wanted θ_1 to positively drive θ_2. While $\frac{-\pi}{4} \leq \theta_1 \leq 0$, cos($\theta_1$) is positive and increasing so θ_1 pushes θ_2 forward with increasing strength. When $0 \leq \theta_1 \leq \frac{\pi}{2}$, cos($\theta_1$) is still positive, but decreasing, so it pushed θ_2 forward with decreasing magnitude. Likewise, θ_2 positively drives $_3$. While $\frac{-\pi}{3} \leq \theta_2 \leq 0$, cos($\theta_2$) is positive and increasing so it is driving θ_3 forward with increasing magnitude, but θ_3 is negative so θ_3 is being pulled to 0. When $0 \leq \theta_2 \leq \frac{\pi}{2}$, cos($\theta_2$) is positive and decreasing so it is still pulling θ_3 closer to 0, but more slowly.

The internal force represents the body's push to move through a stride. I decided on a decaying exponential as I wanted an initially strong force that then decreased through the stride so I settled on f_{int}=10e^{-t}. This made it as though there was a strong initial push from the body and then the "muscle" energy pulled the leg through with decreasing magnitude.

Next, I had to determine the initial conditions of my equations. I chose the following as my initial conditions:

$$\begin{aligned}
\theta_1(0) &= -\frac{\pi}{4} &(23)\\
\dot{\theta}_1(0) &= 0 &(24)\\
\theta_2(0) &= -\frac{\pi}{3} &(25)\\
\dot{\theta}_2(0) &= 0 &(26)\\
\theta_3(0) &= \frac{\pi}{6} &(27)\\
\dot{\theta}_3(0) &= 0 &(28)
\end{aligned}$$

I wanted all the velocities to start at zero so the stride would be from a standing position so all of the even entries – which correspond to the velocities of the angles are zero. For the initial conditions of the angles, we refer back to the Fig. 5. We want to make sure that our initial conditions match our reference. So, we see that θ_1 starts at about a 45 degree angle backwards so in radians we set that to -$\pi/4$. We see θ_2 is open to a slightly less than 90 degree angle so we set θ_2 = -$\pi/3$. We set this to negative because of the same reasoning applied to θ_1. Finally, we see that θ_3 has the opposite reference point and will start positive. We also note that it seems to be open to less than 45 degrees, so we set θ_3 to be $\pi/6$.

It was also important to remember that the equations that govern this model are the equations of a pendulum. This means there are certain constraints we must consider to make this a fair representation of a biological system. The first condition we had to set was in regards to θ_1. Normally, a pendulum swings through its arc, but, if we let θ_1 swing all the way through, we get what Dr. Catlla and I called "no kneecap-itis". This means that the shin would extend past the thigh and we would get something like Fig. 7, where the knee cap basically disappears. To combat this, we added the condition that when θ_2 is greater than 0, then θ_1 must be less than or equal to θ_2. However, once I ran my first simulation, I noticed that this condition was already being met since the internal force was not great enough to push the bottom pendulum (the shin) ahead of the upper pendulum (the thigh).

Additionally, a pendulum will swing back and forth, so I needed to make a logical choice of how to define a single stride. If I ran the simulation for too long, we would see this swinging back behavior but I run it for too small a time, then we lose valuable data. Again, based on Fig. 5 and Fig. 9, I decided to define a stride as from heel strike to midstance, i.e. $\theta_3 = 0$.

Finally, we were ready to solve our equations using Matlab's built in ODE solver, ODE45, to numerically solve these equations. ODE45 is an excellent first order ODE solver so we wrote our second order equations as a system of six first order ODEs.

$$v_1 = \dot{\theta_1} \tag{29}$$

$$v_2 = \dot{v_1} = f_{int} - \omega^2 sin\theta_1 \tag{30}$$

$$v_3 = \dot{\theta_2} \tag{31}$$

$$v_4 = \dot{v_3} = f(\theta_1) - \omega^2 sin\theta_2 \tag{32}$$

$$v_5 = \dot{\theta_3} \tag{33}$$

$$v_6 = \dot{v_5} = f(\theta_2) - \omega^2 sin\theta_3 \tag{34}$$

where $f(\theta)$ is of the form: Acos(Bθ). The full code is provided in Appendix A.

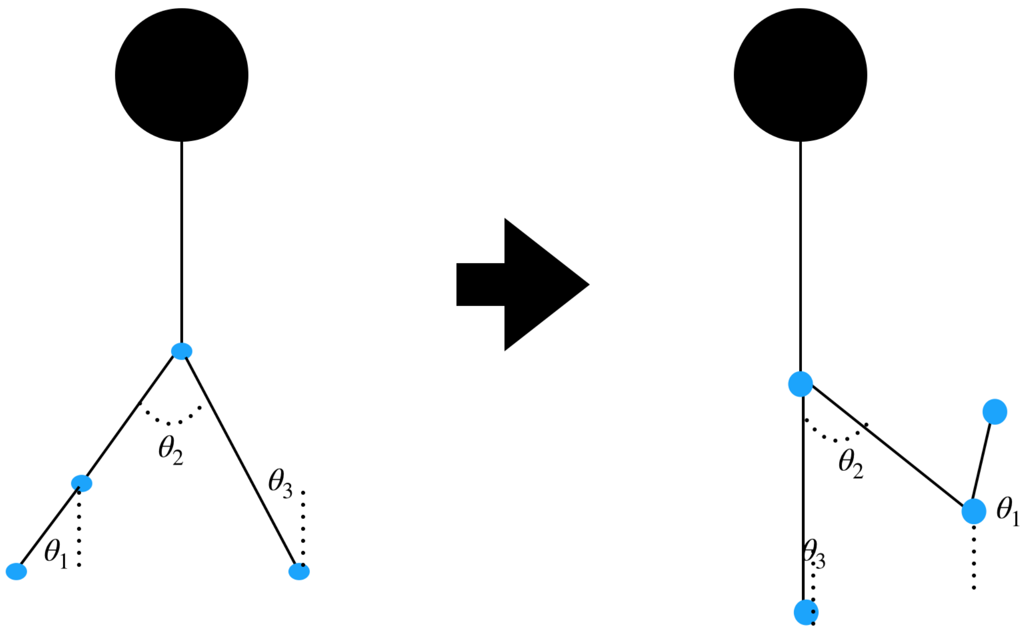


Figure 7: An illustration of θ_1 surpassing $\frac{\theta_2}{2}$

7 Results

7.1 Numerical Methods

I solved the model's equations using Matlab's built-in ODE45 solver, which is a 4th order Runge-Kutta with a 5th order to determine variable time-stepping. What this means is that Matlab estimates the solution to the differential equation with fourth order error. Then, it runs the estimation again, except this time, with fifth order error, and if the two values are close enough, it accepts the fourth order estimate. If they differ by too much, Matlab will repeat the process with a smaller timestep.

We explored two cases numerically. The first was when the internal force is applied to θ_1, which corresponds to a non-prosthetic stride pushing off the foot. The second was the internal force on θ_2, illustrating a stride of an individual with a transtibial prosthetic and the muscle force comes from a lifting of the thigh.

7.2 Internal Force on θ_1

We are interested in how the angles depicted in Fig. 5 change over the course of a single stride shown in the left column of Fig. 8. For the first few time-steps, we see no change in the angles because the initial force is not immediately enough to move the leg. At the start of the stride, we see the internal force on θ_1 pushing θ_2 (Fig. 9), and we see no change in θ_2. This is because the internal force shows in θ_1 first and then in θ_2 due to our coupling term. As θ_1 approaches the middle of the stride, it will increase to become the foot that will be planted. Conversely, as θ_2 moves from the starting to the middle position, it approaches zero as the two legs are moving closer to one another. The angle, θ_3, sees no initial change and then once θ_2 starts approaching zero, θ_3 starts to decrease.

As we move from the middle of the stride to the end of the stride we see in Fig 9 that θ_1 stays below θ_2. We then see θ_1 rounding off as it reaches the end stage, which makes sense because if θ_1 kept increasing, the leg would plant at an awkward angle. As we move from the middle to the end stage, we see θ_2 increases as the legs are moving apart in the opposite direction which sends θ_2 towards $\pi/3$. Finally, as we move from the middle stage to the end stage, we see θ_3 still decreasing towards zero so that it is perpendicular to the ground when our stride ends.

In the phase portrait of θ_1 vs θ_2 in Fig. 9, we see θ_1 will increase while θ_2 does, but always behind it. We also see the curve looking like it is about to loop back around which makes sense as the angles would be resetting so we could stitch together multiple strides.

We consider internal force applied on θ_1 as the stride of a non-prosthetic walker. The force is coming from pushing of the foot and propels the calf forward and then the thigh and then the hips.

7.3 Internal Force on θ_2

We now focus on how the angles depicted in Fig. 5 change over the course of a single stride when the internal force is applied to θ_2. This is illustrated in the left column of Fig. 10. For the first few time-steps we see no change in the angles because the initial force is not immediately enough to move the leg. However, as θ_1 approaches the middle of the stride, it will increase to become the foot that will be planted. Conversely, as θ_2 moves from the starting to the middle position, it approaches zero as the two legs are moving closer to one another. Angle θ_3 sees no initial change and then once θ_2 starts approaching zero, θ_3 starts to decrease.

As we move from the middle of the stride to the end of the stride, we see form Fig. 10 that θ_1 stays below θ_2. We then see θ_1 rounding off as it reaches the end stage, which makes sense because if θ_1 kept increasing, the leg would plant at an awkward angle. As we move from the middle to the end stage, we see θ_2 increases as the legs are moving apart in the opposite direction which sends θ_2 towards $\pi/3$. Finally, as we move from the middle stage to the end stage, we see θ_3 still decreasing towards zero so that it is perpendicular to the ground at the end of a stride.

In the phase portrait of θ_1 vs θ_2 in Fig. 11, we see θ_1 will increase while θ_2 does, but always satisfying $\theta_2 \geq \theta_1$. We also see the curve looking more linear than in the case of the internal force on θ_1. This is because the magnitude of θ_2 is greater when the internal force is applied to θ_2. However, if we ran this a little further it would loop back around which makes sense as the angles would be resetting so we could stitch together multiple strides.

We can consider internal force applied on θ_2 as the stride of a prosthetic walker. The force is coming from the hips and propels the swing leg forward, pulling the calf through.

7.4 Internal Force on θ_3

We originally thought we would check how a stride changes with the internal force applied to each angle. However, we realized that the internal force being applied to only θ_3 was not physically meaningful. Except perhaps for a transfemoral amputation which our model does not apply to. The force only being applied to θ_3 really means that all of the "push" of a step comes from one's standing leg and no effort is produced from the swing leg. This is synonymous to there being no muscular contribution from the swing leg, which is not realistic. Thus, we decided to focus on the case of internal force on θ_1 and θ_2 only.

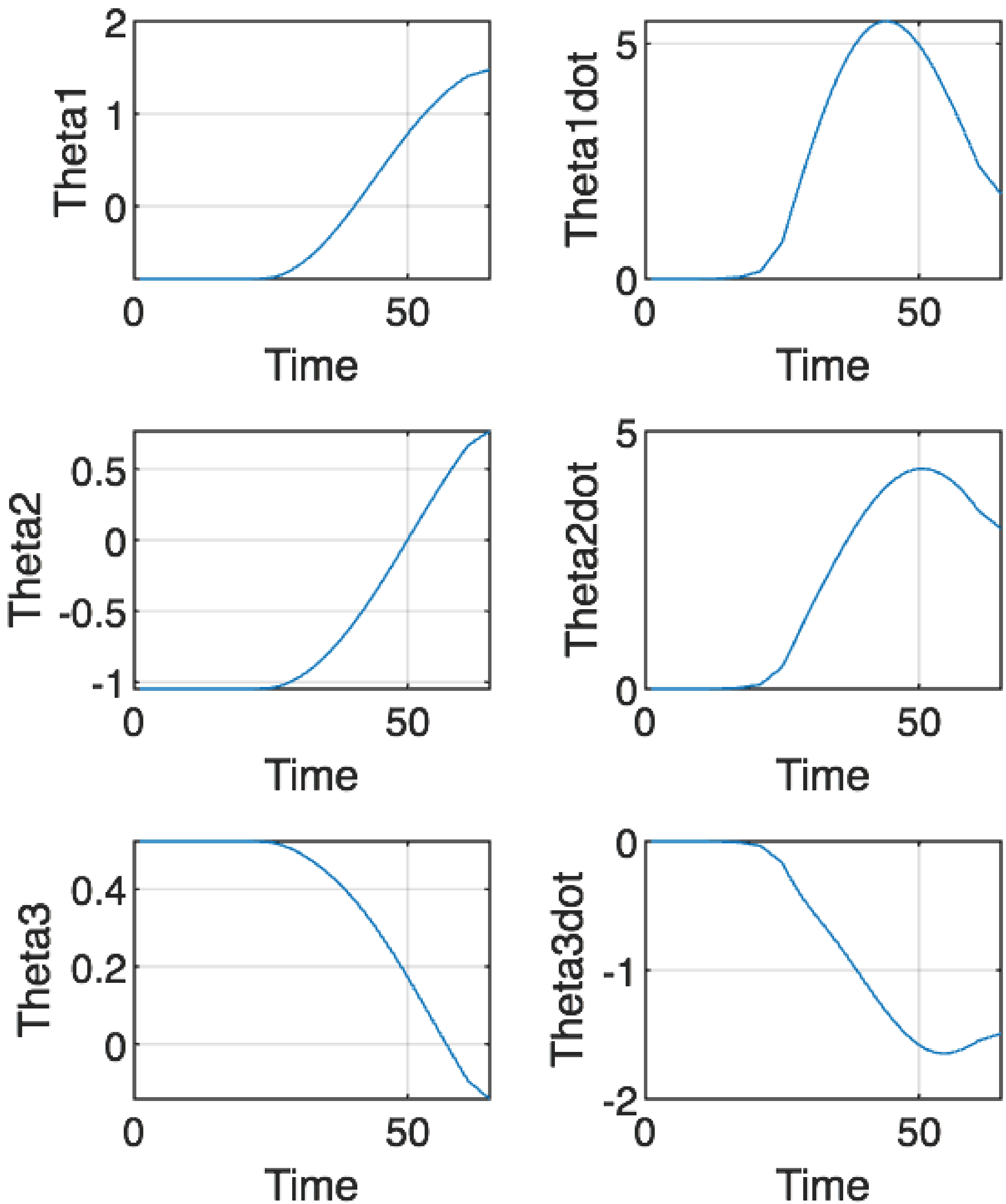
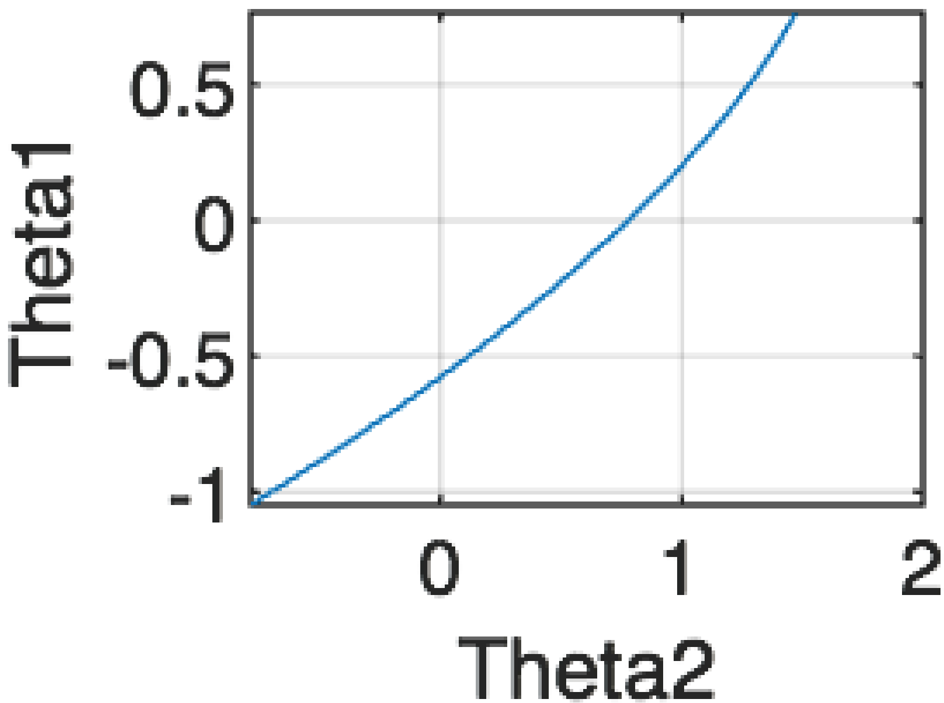
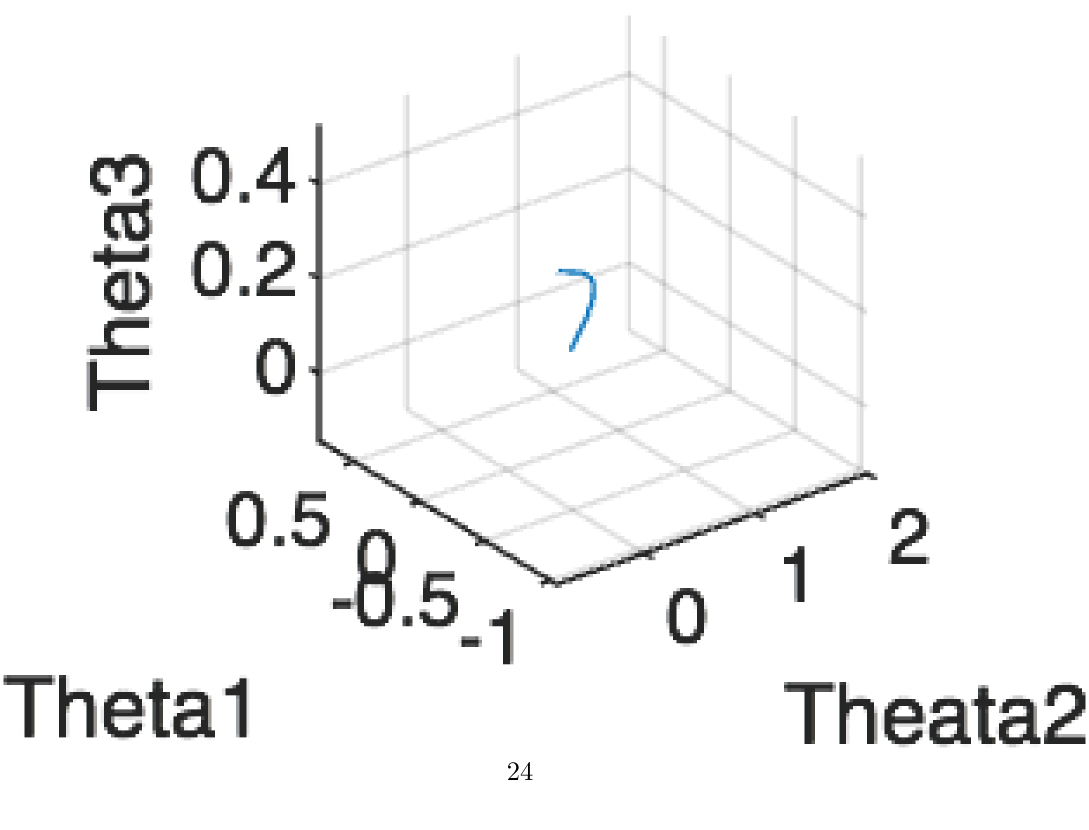


Figure 8: These graphs of the angles were produced from my Matlab code with the internal force being applied to the foot (θ_1) with an initially large magnitude that decreases over time.


0.5
0
-0.5
-1
Theta1
0
1
2
Theta2


Theta3
0.4
0.2
0
0.5
0
-0.5
-1
Theta1
0
1
2
Theata2

7.5 Kinetic Energy Computation

Next, we were interested in how overall kinetic energy changes, depending on the location of the internal force. To calculate the total kinetic energy, we took the kinetic energy on each angle – where the velocity is the angular velocity – and added them together. It is important to note that we used a ratio of the masses rather than specific masses in our equations. The ratio was as follows, m_1=.4, m_2=.6 and m_3=1. At first we focused on the peak kinetic energies, however, we later decided to look at the differences in the integral of kinetic energy as we want to compute the energy over an entire stride. To find the integral of kinetic energy, I used the built in Matlab integral estimator TRAPZ which estimates the area under the curve with trapezoids.

7.6 Comparison of f_{int} on θ_1 and θ_2

Again, we can think of the internal force being applied to θ_2 as someone with a below the knee prosthetic using different muscle groups to propel their stride. In this case, we can compare this to transtibial gait where the stride is propelled by the thigh rather than the ankle. Traumatic transtibial gait has about a 25 percent energy increase compared to non-prosthetic gait and vascular transtibial has a 40 percent increase [6].

When the internal force was changed from being applied to θ_1 to θ_2, there were no obvious quantitative changes but there were qualitative differences. The peak of the overall kinetic energy increased when the internal force was applied to θ_2, as seen in Fig. 12. This makes sense because we consider the kinetic energy as the total energy expenditure of the system. So, we expect there to be a greater energy expenditure with the same force in the prosthetic case compared to the non-prosthetic case.

I was interested in the percent diffence between these two energy values which would avoid issues with units in our non-dimensionalized equations. I took the absolute value of the difference between the integral of kinetic energy on θ_1 and θ_2, and divided by the integral of kinetic energy on θ_1. This is the equation typically used for percent error, but in this case it actually represents the percent difference between the baseline of non-prosthetic gait and prosthetic gait. This calculation gave me a value of about 17 percent (17.74). The literature found a 20 or 40 percent increase in energy for transtibial gait compared to non-prosthetic gait, so our model qualitatively captures this trend [6]. Additionally, it makes sense that our percent difference would be lower as we are essentially modeling a "skeleton" with a push. Thus, there are more components affecting the kinetic energy in real life compared to our model which we would expect to increase the energy. Since, our model accurately describes the trends for energy changes in gait, I can now investigate what

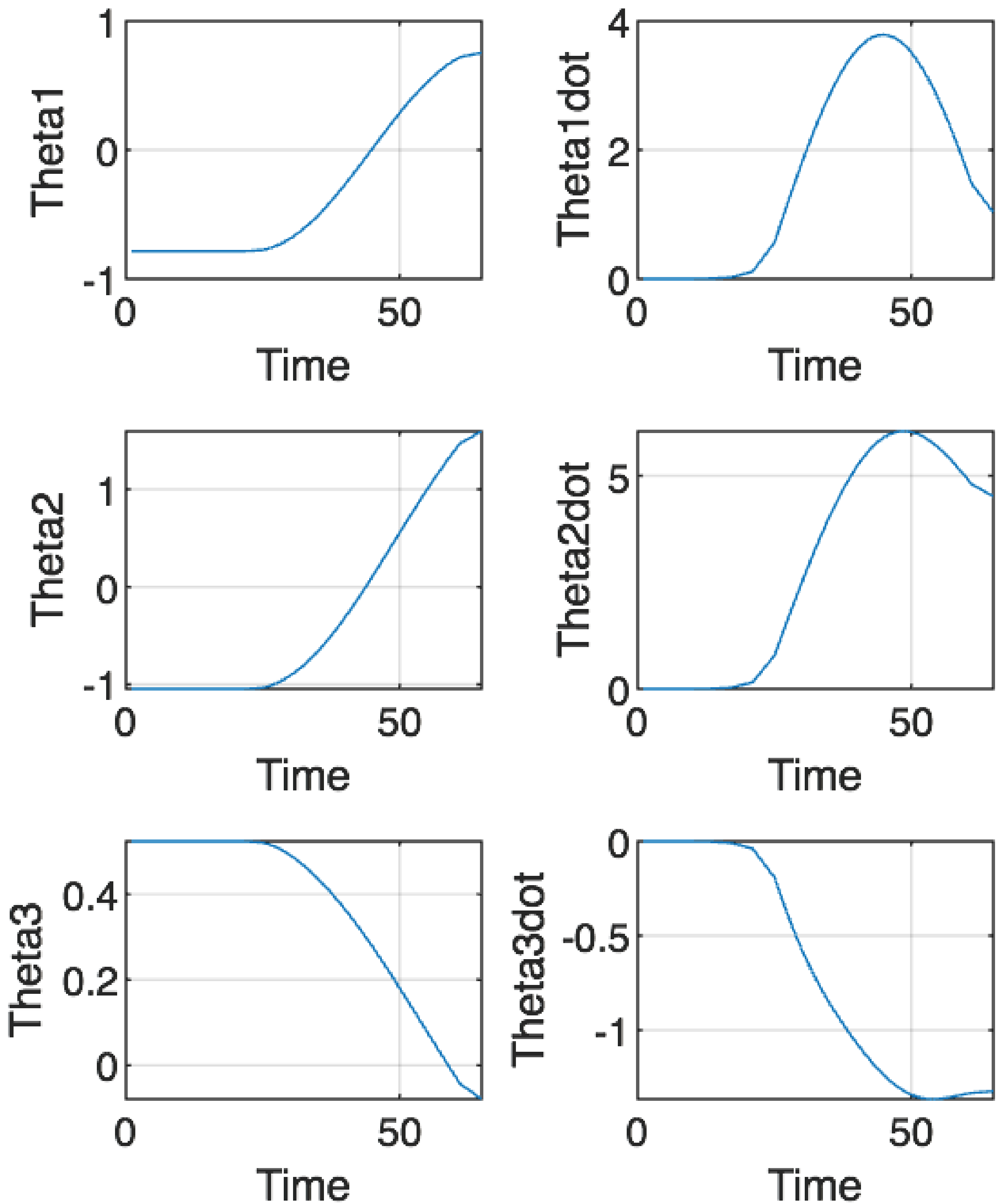


Figure 10: These graphs of the angles were produced from my Matlab code with the internal force being applied to the thigh (θ_2) with an initially large magnitude that decreases over time.

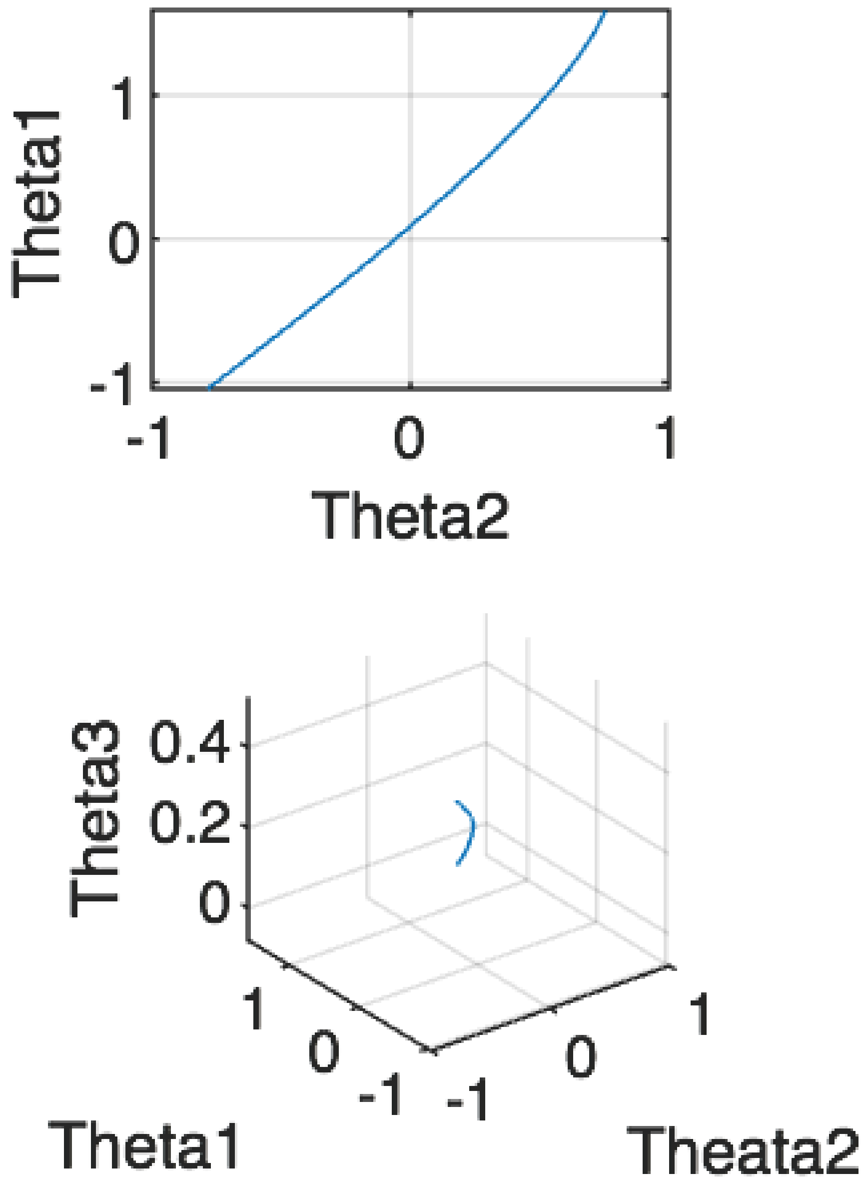


Figure 11: The phase portraits were produced from my Matlab code with the internal force being applied to the thigh (θ_2) with an initially large magnitude that decreases over time.

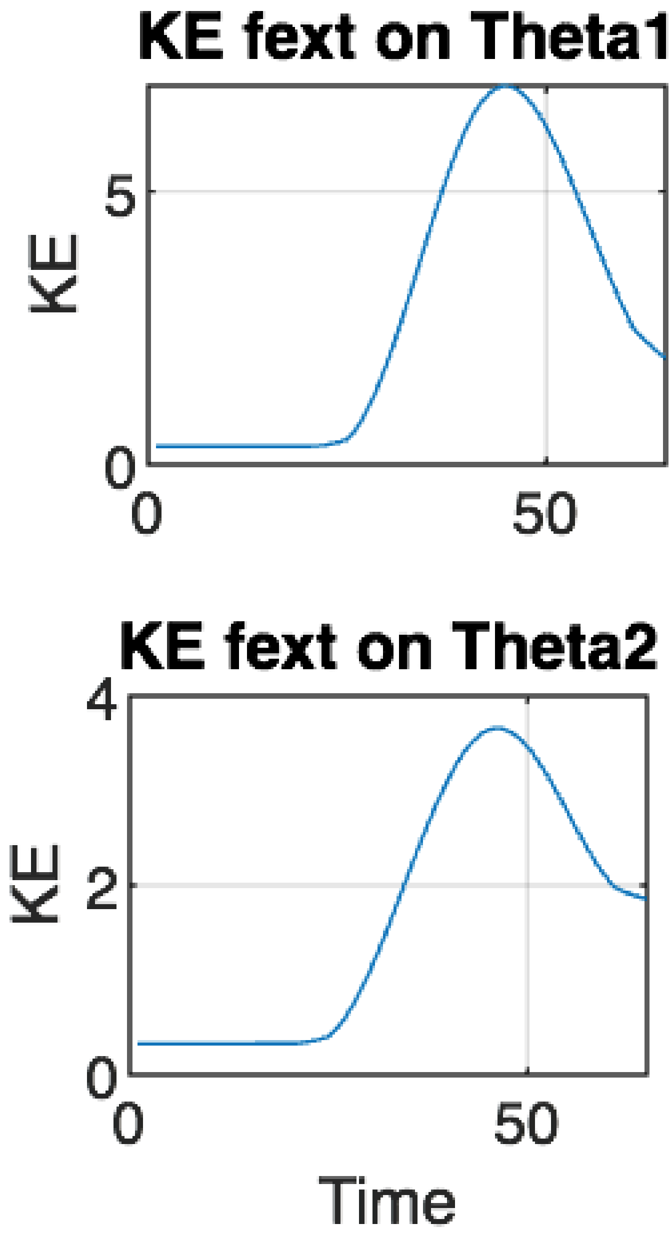


Figure 12: A comparison of the overall kinetic energy of the system with the internal force on θ_1 and θ_2

changes to a prosthetic can help lessen the energy increase needed for prosthetic gait.

7.7 Altering the Prosthetic

I finally get to explore what initially drew me to this project – what can we alter about a prosthetic to make walking easier? I considered what would happen if the mass of the prosthetic was decreased. To do this, I had to decrease the ratio of the shin mass (m_1) to the standing leg (m_3) and the thigh (m_2), but keep the ratio of the thigh to the shin, all while keeping the sum of the masses equal to two. Thus, I had to satisfy the following equations:

$$m_2 = .6m_3 \tag{35}$$

$$m_1 + m_2 + m_3 = 2 \tag{36}$$

$$\frac{m_1}{m_2} < .67. \tag{37}$$

These equations are satisfied with m_1=.24, m_2=.66 and m_3=1.1. When we decrease the mass of the prosthetic, we see the percent energy difference decrease between the non-prosthetic and prosthetic stride from 17 percent to 13.6 percent. So, our model qualitatively expressed the percent energy change when the mass of the prosthetic is decreased.

8 Conclusion

In this paper, we have created a model to describe a single stride. The model, like many others in the literature, is based on a pendulum, but we derived this model ourselves to allow for exploring the effect of a prosthetic [8]. In the creation of our model, we assumed that the swing leg has the prosthetic and that the internal force as a decaying exponential represents the "effort" put in by an individual to move through a single stride. By changing which angle is being forced, we can mimic prosthetic and non-prosthetic cases.

Our simple model accurately represents the angles moving through a stride for both a non-prosthetic and prosthetic individual. Additionally, our results regarding kinetic energy mathematically illustrated the energetic difference between strides. From there we finally asked questions about prosthetic development and what actually influences the energy requirements of a stride: we saw that increasing the velocity of the prosthetic increased the overall kinetic energy, thus decreasing the percent energy difference in prosthetic and non-prosthetic gait. Finally, we saw the addition of an extra force in the prosthetic decreases the percent energy difference in prosthetic and non-prosthetic gait. This model, while simple, seems to be a good qualitative representation of a single stride.

8.1 Further Work

Future work for this project would include adding more parameters into the model to get more qualitative results about the development of prosthesis. Could the model be modified so that muscular activity plays a large role in the results? I would also like to explore different formulas for the internal force being applied and how that alters the results. Finally, I would like to see what happens in the non-prosthetic stride if the internal force is applied to all of the angles but in a specific ratio. One other change we could consider would be to redefine θ_2 to be with respect to the vertical, as θ_1 is, instead of with respect to the standing leg. This could allow us to compute the kinetic energy of just the swing leg, which we think may make for a more accurate comparison.

A Appendix

This appendix contains the code I wrote to numerically solve our model. In appendix A1, we see the code I wrote for the non-prosthetic case where the f_{int} is applied to θ_1. In appendix A2, we see the code I wrote for the prosthetic case where the f_{int} is applied to θ_2.

A.1 Force on θ_1 Code

This contains the code for the case where the internal force is applied to θ_1: [t,vector] = ode45(@f,[0 .6],[-(pi/4) 0 -(pi/3) 0 (pi/6) 0]);

m1=.4; m2=.6; m3=1; l1=.4; l2=.6; l3=1;

KE1=.5*m1*vector(:,2).^{2};$KE2 = .5 * m2 * vector(:,4).^2; KE3 = .5 * m3 * vector(:,6).^2$;

KEtotal=KE1+KE2+KE3;

trapz(KEtotal)

subplot(3,3,1) plot(vector(:,1)) ylabel('Theta1') xlabel('Time') grid on

subplot(3,3,2) plot(vector(:,2)) ylabel('Theta1dot') xlabel('Time') grid on

subplot(3,3,4) plot(vector(:,3)) ylabel('Theta2') xlabel('Time') grid on

subplot(3,3,5) plot(vector(:,4)) ylabel('Theta2dot') xlabel('Time') grid on

subplot(3,3,7) plot(vector(:,5)) ylabel('Theta3') xlabel('Time') grid on

subplot(3,3,8) plot(vector(:,6)) ylabel('Theta3dot') xlabel('Time') grid on

subplot(3,3,3) plot(KEtotal) title('KE fint on Theta1') ylabel('KE') xlabel('Time') grid on

subplot(3,3,6) plot(vector(:,1),vector(:,3)) ylabel('Theta1') xlabel('Theta2') grid on

subplot(3,4,12) plot3(vector(:,1),vector(:,3),vector(:,5)) ylabel('Theta1') xlabel('Theata2') zlabel('Theta3') grid on

function vdot=f(t,vector) if vector(1)¿0 vector(1)¡(vector(3)/2); end

g=9.8;

m1=.4; m2=.6; m3=1; l1=.4; l2=.6; l3=1; omega1 =sqrt(g/l1); omega2 =sqrt(g/l2); omega3 = sqrt(g/l3); fint = 10*exp(-1*t);

v1=vector(2); v2=fint-omega1$^2 * sin(vector(1))$;

v3=vector(4); v4=cos(vector(1))-omega2$^2 * sin(vector(3))$;

v5=vector(6); v6=-cos(vector(3))-omega3$^2 * sin(vector(5))$;

vdot=[v1; v2; v3; v4; v5; v6]; end

A.2 Force on θ_2 Code

This contains the code for the case where the internal force is applied to θ_2:

[t,vector] = ode45(@f,[0 .6],[-(pi/4) 0 -(pi/3) 0 (pi/6) 0]);

m1=.4; m2=.6; m3=1; l1=.4; l2=.6; l3=1;

KE1=.5*m1*vector(:,2).$^2; KE2 = .5 * m2 * vector(:,4).^2; KE3 = .5 * m3 * vector(:,6).^2$;

KEtotal=KE1+KE2+KE3;

trapz(KEtotal)

subplot(3,3,1) plot(vector(:,1)) ylabel('Theta1') xlabel('Time') grid on

subplot(3,3,2) plot(vector(:,2)) ylabel('Theta1dot') xlabel('Time') grid on

subplot(3,3,4) plot(vector(:,3)) ylabel('Theta2') xlabel('Time') grid on

subplot(3,3,5) plot(vector(:,4)) ylabel('Theta2dot') xlabel('Time') grid on

subplot(3,3,7) plot(vector(:,5)) ylabel('Theta3') xlabel('Time') grid on

subplot(3,3,8) plot(vector(:,6)) ylabel('Theta3dot') xlabel('Time') grid on

subplot(3,3,3) plot(KEtotal) title('KE fint on Theta2') ylabel('KE') xlabel('Time') grid on

subplot(3,3,6) plot(vector(:,1),vector(:,3)) ylabel('Theta1') xlabel('Theta2') grid on

subplot(3,4,12) plot3(vector(:,1),vector(:,3),vector(:,5)) ylabel('Theta1') xlabel('Theata2') zlabel('Theta3') grid on

function vdot=f(t,vector)

g=9.8; l1=.4; l2=.6; l3=1; omega1 =sqrt(g/l1); omega2 =sqrt(g/l2); omega3 = sqrt(g/l3); fint = 10*exp(-1*t);

v1=vector(2); v2=-omega1$^2 * sin(vector(1))$;

v3=vector(4); v4=fint-omega2$^2 * sin(vector(3)) - cos(vector(1))$;

v5=vector(6); v6=-cos(vector(3))-omega3$^2 * sin(vector(5))$;

vdot=[v1; v2; v3; v4; v5; v6]; end

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